UNIVERSIDAD DE CONCEPCION
ESCUELA DE GRADUADOS
CONCEPCION-CHILE

METODO DE ELEMENTOS FINITOS PARA PROBLEMAS DE CORRIENTES INDUCIDAS

Tesis para optar al grado de Doctor
en Ciencias Aplicadas con mención en Ingeniería Matemática

Ramiro Miguel Acevedo Martínez

FACULTAD DE CIENCIAS FISICAS Y MATEMATICAS
DEPARTAMENTO DE INGENIERIA MATEMATICA
2008

MÉTODO DE ELEMENTOS FINITOS PARA PROBLEMAS DE CORRIENTES INDUCIDAS

Ramiro Miguel Acevedo Martínez

Directores de Tesis: Rodolfo Rodríguez y Salim Meddahi.

Director de Programa: Raimund Bürger.

COMISION EVALUADORA

Alfredo Bermudez, Universidade de Santiago de Compostela, España.
Gabriel N. Gatica, Universidad de Concepción, Chile
Pilar Salgado, Universidade de Santiago de Compostela, España.
Alberto Valli, Università degli Studi di Trento, Italia.

COMISION EXAMINADORA

Firma: ______________________________

Gabriel N. Gatica
Universidad de Concepción, Chile.

Firma: ______________________________

Rodolfo Rodríguez
Universidad de Concepción, Chile.

Firma: ______________________________

Pilar Salgado
Universidade de Santiago de Compostela, España..

Fecha Examen de Grado: ______________________

Calificación: ______________________

Concepción–Septiembre de 2008

AGRADECIMIENTOS

Agradezco en primer lugar a mis directores de tesis Rodolfo y Salim, a quienes admiro por sus cualidades humanas y profesionales. La ayuda, disposición, paciencia y constante motivación que me bridaron en el desarrollo de este trabajo son realmente invaluables.

A mi esposa Sandra, por todo el amor y compañia que me ha ofrecido en estos años. Muchas veces, cuando las fuerzas me faltaron, en ella encontré el aliciente indicado. Sin ella no hubiera podido lograrlo.

A mis padres y hermanas, quienes creyeron en mí y que a pesar de la distancia, me hicieron sentir que estaban siempre a mi lado.

A todos los grandes amigos que conocí en Chile, quienes han hecho muy gratas las experiencias vividas. En particular, a mis compañeros de la cabina 6, tanto a los que me recibieron cuando comencé mis estudios, como los que aún quedan realizando su doctorado. Ha sido un enorme placer compartir con ustedes todos estos años.

A los profesores del doctorado, a quienes debo gran parte de mi formación. En especial al profesor Gabriel Gatica, quien siempre logró motivarme en cada conversación que sostuvimos.

Al proyecto MECESUP UCO0406, a CONICYT, a la escuela de graduados de la Universidad de Concepción y a la Universidad del Cauca (Colombia), por el financiamiento de mi doctorado.

A mis padres Ramiro y Candelaria,
A mi esposa Sandra.

Resumen

En esta tesis se analizan algunos problemas de corrientes inducidas y la aproximación de sus soluciones a través del método de los elementos finitos. Inicialmente se estudia una formulación en términos de ciertos potenciales de un problema de corrientes inducidas en *régimen armónico* en un dominio acotado. Se realiza un análisis matemático riguroso de dicha formulación en el que se demuestra que la formulación variacional correspondiente es un problema bien planteado. Además, se demuestra que el esquema discreto que se obtiene con subespacios usuales de elementos finitos, converge de forma óptima.

Posteriormente se aborda un *problema evolutivo* de corrientes inducidas, por medio de una formulación que se obtiene a partir de la introducción de una primitiva temporal del campo eléctrico. Esta formulación permite tratar el caso de materiales ferromagnéticos, cuya relación entre la intensidad y la inducción magnética es tipicamente no lineal. El problema se abarca en tres instancias progresivas: problema lineal en un dominio acotado, problema no lineal en un dominio acotado y problema lineal en todo el espacio.

Las formulaciones obtenidas en los casos correspondientes a un dominio acotado tienen estructura mixta, donde se usa un multiplicador de Lagrange para imponer las restricciones del campo eléctrico en el material no conductor. Se demuestra que dichas formulaciones estan bien planteadas y se proponen esquemas semidiscretos (en espacio) basados en elementos finitos de Nédélec para la variable principal y elementos finitos usuales para el multiplicador, y esquemas completamente discretos a través del método de Euler implícito. Además, se demuestran estimaciones optimas del error de ambos esquemas.

La formulación correspondiente al problema en todo el espacio permite combinar un método de elementos finitos mixto (como el del caso acotado) con un método de elementos de frontera. Este acoplamiento se hace introduciendo una variable en la frontera de un cierto dominio acotado que contiene a las regiones de interés, lo que permite aproximar

dicha variable con un subspacio de elementos finitos usuales sobre la frontera. En este caso también se deducen resultados similares a los obtenidos en el caso acotado.

Contents

Resumen **ix**

1 Introducción **1**

1.1 Motivación 1

1.2 Preliminares y discusión bibliográfica 3

1.2.1 Ecuaciones de Maxwell 3

1.2.2 Modelo de corrientes inducidas 7

1.3 Resultados obtenidos y organización de la tesis 10

2 Analysis of a potential formulation for the time-harmonic eddy current problem in a bounded domain **15**

2.1 Introduction 15

2.2 Eddy current problem 17

2.3 The $\boldsymbol{A}, V - \boldsymbol{A} - \psi$ potential formulation 20

2.4 Variational formulation. Existence and uniqueness of solution 24

2.5 Numerical approximation 29

2.6 Conclusions 33

3 An $\boldsymbol{E}$-based mixed formulation for a time-dependent eddy current problem **35**

3.1 Introduction 35

3.2 Preliminaries 37

3.3 Variational formulation 39

3.4 Existence and uniqueness 43

3.5 Analysis of the semi-discrete scheme. 48

3.5.1 Error estimates 52

3.6 Analysis of a fully-discrete scheme. 56
3.6.1 Error estimates 57
3.7 Conclusions. 60

4 Numerical treatment of a nonlinear magnetic field time-dependent eddy current problem 61
4.1 Introduction 61
4.2 Variational formulation 62
4.3 Existence and uniqueness of weak solutions 65
4.4 Analysis of the semi-discrete scheme 69
4.4.1 Error estimates 71
4.5 Analysis of a fully-discrete scheme. 74
4.5.1 Error estimates 76

5 A mixed-FEM and BEM coupling for a time-dependent eddy current problem 79
5.1 Introduction 79
5.2 Preliminaries 81
5.2.1 Tangential differential operators and traces 82
5.2.2 Basic spaces for time dependent problems 85
5.3 The model problem 86
5.4 A mixed FEM-BEM coupling variational formulation 89
5.4.1 The variational formulation in Ω 89
5.4.2 The symmetric FEM-BEM coupling. 91
5.5 Existence and uniqueness. 93
5.6 Analysis of the semi-discrete scheme 102
5.6.1 Error estimates. 107
5.7 Analysis of a fully-discrete scheme. 112
5.7.1 Error estimates. 113

6 Conclusiones y trabajo futuro 119
6.1 Conclusiones 119
6.2 Trabajo futuro 120

Bibliography **122**

Chapter 1

Introducción

1.1 Motivación

En 1873 Maxwell fundó la teoría moderna del electromagnetismo con la publicación de su obra *Treatise on Electricity and Magnetism*, en la cual se formularon las ecuaciones que hoy en dia llevan su nombre. El comportamiento de un campo electromagnético está gobernado por la Ecuaciones de Maxwell, las cuales sintetizan largos años de resultados experimentales en los campos de Electricidad y Magnetismo, en los que resaltan los trabajos de Coulomb, Gauss, Ampere y Faraday. En particular, predicen que los campos eléctricos variables generan campos magnéticos y que, recíprocamente los campos magnéticos variables inducen corrientes eléctricas. Este fenomeno de inducción electromagnética juega un papel fundamental en el diseño adecuado de los dispositivos electrónicos tales como radios, televisores, computadores y hornos de microondas

Debido a lo anterior, desde la formulación de la teoría electromagnética, científicos e ingenieros han buscado la solución exacta de los problemas de contorno que resultan a partir de las ecuaciones de Maxwell. Inicialmente todos los esfuerzos se centraron en determinar estas soluciones a través de métodos análiticos, con los cuales en muchos casos no se obtuvieron resultados satisfactorios. En consecuencia, con el auge de los métodos computacionales, los métodos numéricos resultaron ser una excelente alternativa, lo que ha llevado a que en la actualidad exista un enorme interés por parte de la comunidad de ingeniería y de matemática aplicada en realizar simulaciones del fenómeno electromagnético. Además, por parte de la matemática, recientemente ha crecido el interés en entender las propiedades matemáticas de las ecuaciones de Maxwell que resulten relevantes para su análisis numérico. Evidencia de este interés es el creciente número de articulos y libros dedicados al tema que se encuentran en la literatura actual especializada. Como una pequeña muestra de estos textos podemos citar los libros de Bossavit [27], Monk [57] y Silvester & Ferrari [67], los cuales contienen una larga lista de referencias.

En algunos casos es posible usar un modelo simplificado que aproxime en algún sentido las ecuaciones de Maxwell y que pueda resolverse de una forma más eficiente. Esta situación ocurre por ejemplo en problemas relacionados con maquinas que trabajan en bajas frecuencias, donde puede ser usado el llamado *modelo de corrientes inducidas* (*eddy current model* en la literatura inglesa) que se obtiene a partir de las ecuaciones de Maxwell despreciando las corrientes de desplazamiento de la Ley de Ampère (ver, por ejemplo, [27, Capítulo 8]). Desde un punto de vista matemático, este submodelo genera una aproximación razonable de la solución del sistema completo de ecuaciones de Maxwell en el rango de baja frecuencia [13].

Generalmente el problema de corrientes inducidas esta definido en todo el espacio con condiciones de decaimiento en el infinito, por lo cual la técnica más usada para resolver numéricamente dichas ecuaciones consiste en restringir las ecuaciones a un dominio acotado suficientemente grande que contenga la region de interés e imponiendo condiciones de contorno adecuadas sobre su frontera. Entre los distintos métodos numéricos encontrados en la literatura que usan esta técnica, el método de elementos finitos es el más estudiado. Sus principales ventajas son su flexibilidad respecto a la geometría del problema y las abundantes herramientas teóricas con las que se cuenta para el análisis de convergencia. Sin embargo, en la literatura se halla un número importante de articulos que evitan restringir el problema a un dominio acotado, principalmente combinando el método de los elementos finitos con el de elementos de frontera (Métodos BEM-FEM) [26, 28, 29, 30, 46, 55, 56] .

El método de elementos finitos fue introducido en los cálculos de ingeniería eléctrica en 1970 y desde entonces ha sido aplicado a una gran variedad de problemas de electromagnetismo (ver, por ejemplo los libros de Reece & Preston [63] y Silvester & Ferrari [67]), incluyendo entre estos problemas de corrientes inducidas ([4, 5, 22, 23, 52, 58, 59, 69]). De hecho, actualmente el método de elementos finitos es la base de varios códigos comerciales tales como Ansys, Femlab, Flux, Magnet, MSC/Emas, Opera, etc. Una descripción de estos códigos y otras referencias puede estudiarse en [68].

Cuando se trabaja con corriente alterna, la densidad de corriente impuesta presenta una dependencia armónica respecto al tiempo. En este caso los campos eléctrico y magnético también manifiestan este comportamiento, originando el modelo de corrientes inducidas en *régimen armónico* (*time-harmonic eddy current model* en la literatura inglesa). El régimen armónico tiene la ventaja de manifestar un comportamiento elíptico, en contraste con el problema evolutivo original, que en general implica un problema parabólico acoplado con un problema elíptico. Esta ventaja ha llevado a que el problema armónico haya sido ampliamente estudiado (en [20] se presenta una exposición detallada del tema, incluyendo una completa revisión bibliográfica).

En la siguiente sección, presentamos una breve introducción al modelo de corrientes inducidas (incluyendo el caso del regimén armónico), deduciéndolo a partir de las ecuaciones de Maxwell. Así mismo, realizamos una pequeña discusión bibliográfica mostrando las diversas formulaciones que usan el método de elementos finitos para resolver numericamente el problema de corrientes inducidas.

Acerca de la notación usada. En lo que sigue del capítulo y en el resto de la tesis emplearemos letras negritas para denotar vectores y funciones a valores vectoriales de variable escalar o vectorial. En general, dado un campo vectorial

$$\boldsymbol{\mathcal{F}}(\boldsymbol{x},t) = (\mathcal{F}_1(\boldsymbol{x},t), \mathcal{F}_2(\boldsymbol{x},t), \mathcal{F}_3(\boldsymbol{x},t)), \quad \boldsymbol{x} = (x_1,x_2,x_3) \in \mathbb{R}^3,\ t \in \mathbb{R},$$

div y **curl** representan los operadores *divergencia* y *rotacional* respectivamente, que en coordenadas cartesianas estan definidos por

$$\operatorname{div} \boldsymbol{\mathcal{F}} := \frac{\partial \mathcal{F}_1}{\partial x_1} + \frac{\partial \mathcal{F}_2}{\partial x_2} + \frac{\partial \mathcal{F}_2}{\partial x_2}$$

y

$$\mathbf{curl}\, \boldsymbol{\mathcal{F}} := \left(\frac{\partial \mathcal{F}_3}{\partial x_2} - \frac{\partial \mathcal{F}_2}{\partial x_3}, \frac{\partial \mathcal{F}_1}{\partial x_3} - \frac{\partial \mathcal{F}_3}{\partial x_1}, \frac{\partial \mathcal{F}_2}{\partial x_1} - \frac{\partial \mathcal{F}_1}{\partial x_2} \right).$$

Además de estos dos operadores, también usaremos el operador *gradiente* de funciones escalares $f(\boldsymbol{x},t)$, que denotaremos por **grad** y que se define por

$$\mathbf{grad}\, f := \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right).$$

1.2 Preliminares y discusión bibliográfica

1.2.1 Ecuaciones de Maxwell

Los fenómenos electromagnéticos se describen a través de cuatro campos vectoriales que dependen de la posición espacial $\boldsymbol{x} = (x_1,x_2,x_3) \in \mathbb{R}^3$ y del tiempo $t \geq 0$:

- la intensidad de campo eléctrico $\boldsymbol{\mathcal{E}}$,
- el desplazamiento eléctrico $\boldsymbol{\mathcal{D}}$,
- la intensidad de campo magnético $\boldsymbol{\mathcal{H}}$ y
- la inducción magnética $\boldsymbol{\mathcal{B}}$.

Los campos anteriores son generados por dos tipos de fuentes: *cargas eléctricas* y flujos de carga eléctrica variable llamados *corrientes*. La distribución de cargas es dada por una función escalar ρ que representa la *densidad de carga eléctrica*, mientras que las corrientes son descritas por una función vectorial de *densidad de corriente* $\boldsymbol{\mathcal{J}}$.

La relación que existe entre los campos $\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{D}}, \boldsymbol{\mathcal{H}}$ y $\boldsymbol{\mathcal{B}}$, y las fuentes ρ y $\boldsymbol{\mathcal{J}}$, está dada por un conjunto de ecuaciones llamadas *ecuaciones de Maxwell*, que son válidas en todo punto $\boldsymbol{x} \in \mathbb{R}^3$ y en todo tiempo $t \geq 0$. Las ecuaciones de Maxwell en su forma diferencial son las siguientes ([48]):

$$\frac{\partial \boldsymbol{\mathcal{D}}}{\partial t} - \mathbf{curl}\, \boldsymbol{\mathcal{H}} = -\boldsymbol{\mathcal{J}}, \tag{1.1}$$

$$\frac{\partial \boldsymbol{\mathcal{B}}}{\partial t} + \mathbf{curl}\, \boldsymbol{\mathcal{E}} = \mathbf{0}, \tag{1.2}$$

$$\operatorname{div} \boldsymbol{\mathcal{D}} = \rho. \tag{1.3}$$

$$\operatorname{div} \boldsymbol{\mathcal{B}} = 0. \tag{1.4}$$

La ecuación (1.1) es llamada la *ley de Ampère-Maxwell*, que relaciona el desplazamiento eléctrico con la variación en el tiempo del campo magnético. La ley de Ampère-Maxwell coincide con la *ley de Ampère* salvo por el término adicional $\dfrac{\partial \boldsymbol{\mathcal{D}}}{\partial t}$ introducido por Maxwell. Este término adicional se conoce en la literatura como *corrientes de desplazamiento*.

La ecuación (1.2) es la *ley de Faraday* y relaciona el campo eléctrico con la variación respecto al tiempo de la inducción magnética.

La tercera ecuación (1.3) es la *ley de Gauss* y físicamente significa que el flujo de la inducción eléctrica a través de una superficie cerrada es igual a la carga neta encerrada dentro de dicha superficie, lo cual implica que las lineas de carga eléctrica empiezan y terminan con cargas eléctricas.

Por último, la ecuación (1.4) es conocida como *ley de Gauss del magnetismo* y afirma que el flujo de la inducción magnética a través de cualquier superficie cerrada es nulo. Una consecuencia natural de esta ley es la inexistencia de polos magnéticos aislados.

Podemos notar que si (1.4) se cumple para $t = 0$, entonces a partir de (1.2), se puede deducir (1.4) para todo $t > 0$. Además, si (1.3) se verifica para $t = 0$ y se asume que la *carga se conserva*, es decir que $\boldsymbol{\mathcal{J}}$ y ρ estan relacionadas a través de la igualdad

$$\operatorname{div} \boldsymbol{\mathcal{J}} + \frac{\partial \rho}{\partial t} = 0,$$

entonces la ley de Ampère implica (1.3) para todo $t > 0$ (ver [57, Capítulo 1]). Debido a esto, a partir de ahora consideraremos como ecuaciones (fundamentales) del sistema de Maxwell, las leyes (1.1) y (1.2).

Para obtener un sistema cerrado a partir de las ecuaciones de Maxwell (1.1) y (1.2), requerimos de información adicional que vincule los distintos campos entre si. Esta información es dada a través de las *leyes constitutivas*:

$$\begin{aligned} \boldsymbol{\mathcal{B}} &= \mu \boldsymbol{\mathcal{H}}, \\ \boldsymbol{\mathcal{D}} &= \varepsilon \boldsymbol{\mathcal{E}}, \end{aligned}$$

y por la *ley de Ohm*

$$\boldsymbol{\mathcal{J}} = \boldsymbol{\mathcal{J}}_{\mathrm{d}} + \sigma \boldsymbol{\mathcal{E}},$$

donde

- $\boldsymbol{\mathcal{J}}_{\mathrm{d}}$ es la densidad de la fuente de *corriente aplicada*,
- ε es la *permitividad eléctrica*,
- μ es la *permeabilidad magnética* y
- σ es la *conductividad eléctrica.*

Los parámetros ε, μ y σ dependen del material ocupado por el dominio del campo electromagnético. El caso que más se presenta en las aplicaciones prácticas es en el que dicho dominio está compuesto por diversos materiales (por ejemplo cobre, hierro, aire, etc). En este caso el medio es llamado *no homogéneo.* Los materiales que estudiaremos en esta tesis son no homogéneos *isotrópicos*, es decir aquellos en los que las propiedades del material no dependen de la dirección del campo. En este caso, los parámetros en cuestión son funciones escalares, no negativas y acotadas.

Los parámetros ε y σ dependen de las caractéristicas eléctricas del material, por ejemplo en el caso de materiales *conductores* σ y ε son son funciones positivas que dependen de la posición. Mientras que en materiales no conductores (*dieléctricos*) $\sigma = 0$ y ε es una constante positiva.

Por otra parte, μ depende de las propiedades magnéticas del medio. La dependencia o no del parametro μ de la intensidad del campo magnético, clasifica los materiales no homogéneos isotrópicos en *lineales* y *no lineales*. Los materiales lineales son aquellos en los que μ solo depende de la posición $\boldsymbol{x} \in \mathbb{R}^3$, mientras que en los materiales no lineales (por ejemplo los materiales *ferromagnéticos* como el hierro, cobre, etc), μ depende del modulo del campo magnético $\boldsymbol{\mathcal{H}}$, es decir

$$\mu = \mu(|\boldsymbol{\mathcal{H}}|).$$

Usando las anteriores leyes constitutivas y la ley de Ohm, podemos expresar las ecuaciones de Maxwell solo en términos de los campos de principal interés físico ($\boldsymbol{\mathcal{E}}$ y $\boldsymbol{\mathcal{H}}$), obteniendo el siguiente sistema de ecuaciones:

$$\frac{\partial\left(\varepsilon\boldsymbol{\mathcal{E}}\right)}{\partial t} + \sigma\boldsymbol{\mathcal{E}} - \mathbf{curl}\,\boldsymbol{\mathcal{H}} = -\boldsymbol{\mathcal{J}}_{\mathrm{d}}, \tag{1.5}$$

$$\frac{\partial\left(\mu\boldsymbol{\mathcal{H}}\right)}{\partial t} + \mathbf{curl}\,\boldsymbol{\mathcal{E}} = \mathbf{0}. \tag{1.6}$$

Las ecuaciones (1.5)-(1.6) deben ser complementadas con condiciones iniciales y con una condición de disipación en el infinito, llamada *condición de radiación de Silver-Müller* [60]:

$$\lim_{|\boldsymbol{x}|\to+\infty}\left(\boldsymbol{\mathcal{H}} \times \frac{\boldsymbol{x}}{|\boldsymbol{x}|} - \boldsymbol{\mathcal{E}}\right) = \mathbf{0}. \tag{1.7}$$

Ecuaciones de Maxwell en régimen armónico

Si se desea estudiar la propagación electromagnética en el caso en que las fuentes de corriente y cargas varian sinusoidalmente respecto al tiempo, entonces el problema evolutivo (1.5)-(1.7) se reduce al conocido como sistema de Maxwell en *régimen armónico*. En este caso, las fuentes $\boldsymbol{\mathcal{J}}_{\mathrm{d}}$ y ρ, y los campos $\boldsymbol{\mathcal{E}}$ y $\boldsymbol{\mathcal{H}}$ tienen la forma

$$\begin{aligned} \rho(\boldsymbol{x},t) &= \operatorname{Re}\left[e^{i\omega t}q(\boldsymbol{x})\right], \\ \boldsymbol{\mathcal{J}}_{\mathrm{d}}(\boldsymbol{x},t) &= \operatorname{Re}\left[e^{i\omega t}\boldsymbol{J}_{\mathrm{d}}(\boldsymbol{x})\right], \\ \boldsymbol{\mathcal{E}}(\boldsymbol{x},t) &= \operatorname{Re}\left[e^{i\omega t}\boldsymbol{E}(\boldsymbol{x})\right], \\ \boldsymbol{\mathcal{H}}(\boldsymbol{x},t) &= \operatorname{Re}\left[e^{i\omega t}\boldsymbol{H}(\boldsymbol{x})\right], \end{aligned} \tag{1.8}$$

donde $q : \mathbb{R}^3 \to \mathbb{C}$, $\boldsymbol{J} : \mathbb{R}^3 \to \mathbb{C}^3$, $\boldsymbol{E} : \mathbb{R}^3 \to \mathbb{C}^3$ y $\boldsymbol{H} : \mathbb{R}^3 \to \mathbb{C}^3$ son llamados amplitudes complejas de ρ, $\boldsymbol{\mathcal{J}}$, $\boldsymbol{\mathcal{E}}$ y $\boldsymbol{\mathcal{H}}$ respectivamente. El parámetro real $\omega \neq 0$ es llamado *frecuencia de corriente*. Sustituyendo las expresiones anteriores en el sistema (1.5)-(1.7), obtenemos las ecuaciones de Maxwell en régimen armónico:

$$i\omega\varepsilon\boldsymbol{E} + \sigma\boldsymbol{E} - \mathbf{curl}\,\boldsymbol{H} = -\boldsymbol{J}, \tag{1.9}$$

$$i\omega\mu\boldsymbol{H} + \mathbf{curl}\,\boldsymbol{E} = \mathbf{0}, \tag{1.10}$$

$$\lim_{|\boldsymbol{x}|\to+\infty} \left(\boldsymbol{H} \times \frac{\boldsymbol{x}}{|\boldsymbol{x}|} - \boldsymbol{E} \right) = \mathbf{0}. \tag{1.11}$$

1.2.2 Modelo de corrientes inducidas

En diversas situaciones físicas es posible omitir el término que envuelve el desplazamiento eléctrico $\varepsilon\boldsymbol{\mathcal{E}}$ de la ley de Ampère-Maxwell (1.5). La supresión de este término es razonable cuando la magnitud de las corrientes de desplazamiento $\dfrac{\partial(\varepsilon\boldsymbol{\mathcal{E}})}{\partial t}$ es despreciable respecto al resto de términos de (1.5) (*hipótesis cuasiestática*). En particular, en el régimen armónico, esto es valido si la frecuencia de corriente ω es suficientemente pequeña (ver [27, Capítulo 8]).

El modelo que se obtiene al asumir la hipótesis cuasiestática es llamado *modelo de corrientes inducidas*. En este modelo, además de asumirse la variación de la ley de Ampère-Maxwell antes mencionada, también se sustituye la condición de radiación de Silver-Müller, por ciertas condiciones de decaimiento en el infinito que cumplen los campos $\boldsymbol{\mathcal{E}}$ y $\boldsymbol{\mathcal{H}}$ independientemente. Más precisamente, el problema de corrientes inducidas consite en determinar los campos $\boldsymbol{\mathcal{E}}$ y $\boldsymbol{\mathcal{H}}$ que satisfacen:

$$\mathbf{curl}\,\boldsymbol{\mathcal{H}} - \sigma\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{J}}_{\rm d}, \tag{1.12}$$

$$\frac{\partial(\mu\boldsymbol{\mathcal{H}})}{\partial t} + \mathbf{curl}\,\boldsymbol{\mathcal{E}} = \mathbf{0}, \tag{1.13}$$

$$\boldsymbol{\mathcal{H}}(\boldsymbol{x},t) = O\left(\frac{1}{|\boldsymbol{x}|}\right) \qquad \text{si } |\boldsymbol{x}| \to \infty, \tag{1.14}$$

$$\boldsymbol{\mathcal{E}}(\boldsymbol{x},t) = O\left(\frac{1}{|\boldsymbol{x}|}\right) \qquad \text{si } |\boldsymbol{x}| \to \infty. \tag{1.15}$$

El modelo de corrientes inducidas en régimen armónico se obtiene a partir del sistema anterior, asumiendo que los campos y fuentes tienen la forma (1.8). En este caso el problema consiste en determinar las amplitudes $\boldsymbol{E}$ y $\boldsymbol{H}$ que verifican:

$$\mathbf{curl}\,\boldsymbol{H} - \sigma\boldsymbol{E} \;=\; \boldsymbol{J}_{\mathrm{d}}, \tag{1.16}$$

$$i\omega\mu\boldsymbol{H} + \mathbf{curl}\,\boldsymbol{E} \;=\; \mathbf{0}, \tag{1.17}$$

$$\boldsymbol{H}(\boldsymbol{x}) \;=\; O\left(\frac{1}{|\boldsymbol{x}|}\right) \qquad \text{si } |\boldsymbol{x}| \to \infty, \tag{1.18}$$

$$\boldsymbol{E}(\boldsymbol{x}) \;=\; O\left(\frac{1}{|\boldsymbol{x}|}\right) \qquad \text{si } |\boldsymbol{x}| \to \infty. \tag{1.19}$$

No es razonable esperar que los sistemas (1.12)-(1.15) y (1.16)-(1.19) tengan solución única. Por ejemplo, siendo $\sigma = 0$ en el material dieléctrico, una solución $\boldsymbol{\mathcal{E}}$ de (1.12)-(1.15) puede ser perturbada mediante adición de cualquier gradiente soportado en dicho material que cumpla la condición (1.15) y se obtiene una nueva solución del sistema. Un análisis similar se puede realizar para el problema armónico.

Debido a lo anterior, es necesario imponer condiciones adicionales a los campos, que garanticen existencia y unicidad de solución. Para imponer tales condiciones, es natural que se recurra a propiedades de las soluciones originales del sistema de Maxwell, que no puedan deducirse de (1.12)-(1.15).

A partir de la Ley de Gauss (1.3) en ausencia de carga eléctrica ($\rho = 0$), se obtiene que $\boldsymbol{\mathcal{E}}$ debe verificar

$$\operatorname{div}(\varepsilon\boldsymbol{\mathcal{E}}) \;=\; 0 \qquad\qquad \text{en } \left(\mathbb{R}^3 \setminus \Omega_{\mathrm{c}}\right), \tag{1.20}$$

$$\int_{\Sigma_i} \varepsilon\boldsymbol{\mathcal{E}} \cdot \boldsymbol{n} \;=0 \qquad i = 1, \cdots, I, \tag{1.21}$$

donde Ω_{c} es el dominio (acotado) que ocupa el material conductor, Σ_i ($i = 1, \cdots, I$) son las componentes conexas de su frontera $\Sigma := \partial\Omega_{\mathrm{c}}$ y $\boldsymbol{n}$ es el vector unitario normal exterior a Σ. Agregando estas dos restricciones al problema (1.12)-(1.15), obtenemos el modelo de corrientes inducidas, que es precisamente el tema de estudio de esta tesis.

Si expresamos las restricciones (1.20) y (1.21) en términos de la amplitud $\boldsymbol{E}$ de $\boldsymbol{\mathcal{E}}$, se tiene:

$$\operatorname{div}(\varepsilon\boldsymbol{E}) \;=\; 0 \qquad\qquad \text{en } \left(\mathbb{R}^3 \setminus \Omega_{\mathrm{c}}\right), \tag{1.22}$$

$$\int_{\Sigma_i} \varepsilon\boldsymbol{E} \cdot \boldsymbol{n} \;=0 \qquad i = 1, \cdots, I. \tag{1.23}$$

Así, el modelo de corrientes inducidas en régimen armónico se obtiene agregando estas restricciones al sistema (1.16)-(1.19).

Es bien conocido que el sistema completo (1.16)-(1.19), (1.22)-(1.23) posee una única solución [13, Teorema 3.2]. En esta misma referencia se demuestra que bajo ciertas hipótesis sobre la fuente de corriente, la solución de este sistema genera una aproximación (de segundo orden respecto a la frecuencia) de la solución del sistema de Maxwell completo (1.9)-(1.11) [13, Sección 7]. Un resultado de aproximación similar se tiene para las soluciones de los problemas evolutivos respectivos [13, Sección 8].

Aunque el procedimiento usual para resolver el problema de corrientes inducidas es restringirlo a un dominio acotado que contenga las regiones de interés (dominio conductor y soportes de las fuentes), existen trabajos dedicados a resolverlo en todo el espacio completo, entre los que sobresalen los basados en técnicas BEM-FEM. Por ejemplo, Hiptmair [46] estudia una acoplamiento BEM-FEM para una formulación del problema armónico tomando como variable principal el campo eléctrico $\boldsymbol{E}$, en contraste con la formulación en términos del campo magnético $\boldsymbol{H}$ propuesta por Bossavit [27] para el mismo problema, que posteriormente fue extendida por Meddahi and Selgas, tanto para el régimen armónico [55], como para el problema evolutivo [56].

El uso del método de elementos finitos para resolver el problema de corrientes inducidas implica restringir dicho problema a un dominio acotado Ω, con lo cual deben imponerse condiciones de contorno adecuadas (desde el punto de vista de las aplicaciones) sobre la frontera $\partial\Omega$. Alonso & Valli [10] estudiaron el problema armónico acotado formulándolo en términos del campo eléctrico y demostraron que el problema continuo con condiciones de Dirichlet no homogéneas tiene solución única. Otra formulación para el problema acotado en régimen armónico, basada en el campo magnético fue estudiada por Alonso, Hiptmair & Valli [8].

Como se mencionó antes, el modelo (1.16)-(1.19), (1.22)-(1.23) restringido a un dominio acotado Ω (con condiciones de contorno adecuadas), permite determinar los campos eléctrico y magnético en dicho dominio. Sin embargo, en diversas situaciones solo interesa calcular el campo eléctrico en el material conductor. En este caso, las ecuaciones (1.22)-(1.23) no pueden ser consideradas, ya que solo tienen sentido si se considera $\boldsymbol{E}$ en el material dieléctrico. Además, la ecuación (1.17) es valida solamente en el conductor y por lo tanto la condición de divergencia nula de $\mu\boldsymbol{H}$ (que se obtiene de (1.17)) se verificará solamente en Ω_{c}. En consecuencia, la restricción de divergencia nula de $\mu\boldsymbol{H}$ en todo Ω deberá ser una ecuación adicional al sistema (1.16)-(1.19), en cuyo caso obtenemos el siguiente problema (asumiendo que $\operatorname{supp}\boldsymbol{J}_{\mathrm{d}} \cup \overline{\Omega}_{\mathrm{c}} \subset \Omega$ and $\operatorname{supp}\boldsymbol{J}_{\mathrm{d}} \cap \overline{\Omega}_{\mathrm{c}} = \phi$):

Hallar $\boldsymbol{H}:\Omega\to\mathbb{C}$ and $\boldsymbol{E}:\Omega_{\rm c}\to\mathbb{C}$ tales que

$$\begin{aligned}
\mathbf{curl}\,\boldsymbol{H} &= \sigma\boldsymbol{E} && \text{en } \Omega_{\rm c}, && (1.24)\\
\mathbf{curl}\,\boldsymbol{H} &= \boldsymbol{J}_{\rm d} && \text{en } \Omega\setminus\overline{\Omega}_{\rm c}, && (1.25)\\
i\omega\mu\boldsymbol{H}+\mathbf{curl}\,\boldsymbol{E} &= \mathbf{0} && \text{en } \Omega_{\rm c}, && (1.26)\\
\operatorname{div}(\mu\boldsymbol{H}) &= 0 && \text{en } \Omega\setminus\overline{\Omega}_{\rm c}, && (1.27)\\
\boldsymbol{H}|_{\Omega_{\rm c}}\times\boldsymbol{n}-\boldsymbol{H}|_{\Omega\setminus\overline{\Omega}_{\rm c}}\times\boldsymbol{n} &= \mathbf{0} && \text{en } \Sigma, && (1.28)\\
(\mu\boldsymbol{H})\,|_{\Omega_{\rm c}}\cdot\boldsymbol{n}-(\mu\boldsymbol{H})\,|_{\Omega\setminus\overline{\Omega}_{\rm c}}\cdot\boldsymbol{n} &= 0 && \text{en } \Sigma. && (1.29)
\end{aligned}$$

Diversas formulaciones de elementos finitos para resolver el problema (1.24)-(1.29) restringido a un conjunto acotado han sido propuestas inicialmente por ingenieros. Algunas de éstas se basan en uno de los dos campos principales ($\boldsymbol{E}$,$\boldsymbol{H}$): Albanese & Rubinacci [3], Golias *et al* [51], Rodger & Eastham [66], Tu *et al* [69], Webb *et al* [70, 71, 72]. Algunas formulaciones basadas en $\boldsymbol{E}$ o en $\boldsymbol{H}$ han sido estudiadas en la última decada eesde el punto de vista matemático, donde resaltan los trabajos de Alonso *et al* [7, 8, 9, 10, 11] y Bermúdez *et al* [16, 18, 17, 19].

Otras formulaciones basadas en potenciales escalares y vectoriales, también han sido propuestas por ingenieros: Albertz & Henneberger [4, 5], Bíró [22], Bíró & Preis [23], Golias *et al* [51], Leonard & Rodger [52]. En lo que respecta al análisis matemático de estas formulaciones, el número de trabajos en la literatura es menor comparado con los del primer grupo, podemos citar por ejemplo Alonso *et al* [7], Bíró & Valli [24], Fernandez & Valli [42].

1.3 Resultados obtenidos y organización de la tesis

El objetivo de esta tesis es estudiar algunas formulaciones del problema de corrientes inducidas, que permiten aproximar la solución de dicho problema a través del método de elementos finitos. El estudio abarca tanto el problema evolutivo de corrientes inducidas (Problema (1.12)-(1.15), (1.20)-(1.21)), como el problema en régimen armónico (Problema (1.24)-(1.29)). El problema evolutivo es estudiado inicialmente reducido a un dominio acotado (Capítulos 3 y 4) y finalmente es analizado en todo el espacio (capítulo 5).

En el segundo capítulo se analiza una formulación del problema (1.24)-(1.29) en términos de ciertos potenciales de los campos eléctrico y magnético, introducida por

Leonard & Rodger [52]. Esta formulación plantea calcular un potencial escalar eléctrico en el conductor, un potencial vectorial magnetico en un conjunto que contiene al conductor y al soporte de la fuente de corriente, y un potencial escalar magnético en el complemento de este último conjunto con respecto al conjunto acotado al que fue reducido el problema. A pesar de que hay experimentos numéricos que comparan esta formulación con otras en términos de potenciales y que muestran la eficiencia del método (ver [23]), no hay un análisis matemático en el que se demuestre existencia y unicidad de solución, y convergencia del método de elementos finitos. En el segundo capítulo de esta tesis, se presenta el análisis matemático de este método.

Despues de realizar una deducción formal de la formulación variacional del problema, se procede a demostrar la elipticidad de la forma bilineal asociada a dicha formulación, con lo cual la existencia y unicidad de solución es una consecuencia del Lema de Lax-Milgram. Posteriormente, se deduce el esquema de Galerkin respectivo, que también resulta estar bien planteado y se demuestra que bajo la regularidad necesaria de los potenciales, el método de elementos finitos converge a la solución en una forma óptima.

Una característica de esta formulación es que usa elementos finitos usuales (funciones polinomiales a trozos y continuas), lo que se traduce en facilidad para adaptar códigos ya elaborados para otro tipo de problemas y en una evidente reducción del número de grados de libertad, en relación con el uso de otros tipos de elementos finitos. Para que el potencial vectorial magnético posea la regularidad que hace viable la aproximación por elementos finitos usuales y a su vez esta converja (teóricamente) a la solución del problema, se concluye que el dominio de definición de dicho potencial debe tener la propiedad de que cada una de sus componentes conexas debe ser convexa (ver Sección 2.5). Los resultados de este capítulo dieron origen a la publicación [2]:

- R. Acevedo and R. Rodríguez, *Analysis of the* $\boldsymbol{A}$*,* $V - \boldsymbol{A} - \psi$ *potential formulation for the eddy current problem in a bounded domain*, Electron. Trans. Numer. Anal., 26 (2007), pp. 270–284.

En el capítulo 3 se estudia una nueva formulación para el problema evolutivo (1.12)-(1.15),(1.20)-(1.21) restringido a un dominio acotado, que usa como variable principal una primitiva temporal del campo eléctrico. Las restricciones (1.20)-(1.21) se imponen a través de un multiplicador de Lagrange. La formulación variacional que se obtiene es la de una ecuación parabólica degenerada en forma mixta. La estrategia que se sigue para demostrar que el problema tiene una única solución consiste en obtener una adecuada descomposición del núcleo de la forma bilineal que impone débilmente las restricciones (1.20)-(1.21), lo cual permite descomponer el problema (reducido a dicho núcleo) en dos problemas: uno parabólico clásico y otro elíptico. Posteriormente, la existencia y unicidad del multiplicador se deduce a través de técnicas propias de problemas parabólicos en forma mixta (ver, por ejemplo, [21]).

La semidiscretización espacial del problema, se lleva a cabo usando elementos de Nédélec para la variable principal y elementos finitos usuales para el multiplicador. Se demuestra que el esquema de Galerkin está bien planteado, siguiendo ideas similares a las usadas para demostrar que el problema continuo tiene solución única. Posteriormente, la discretización completa en espacio y tiempo, se realiza manteniendo la misma discretización espacial que en el esquema semidiscreto y un esquema de Euler implícito para la discretización temporal. En este caso, en cada paso de tiempo debe resolverse un problema mixto clásico, cuya existencia y unicidad de solución se demuestra a través de la teoría de Babuška-Brezzi. Finalmente, para ambos esquemas (semidiscreto y completamente discreto) se demuestra convergencia óptima de la aproximación por elementos finitos. El análisis además nos permite deducir estimaciones del error para la aproximación de las variables físicas más relevantes: las corrientes inducidas en el material conductor y la inducción magnética en todo el dominio acotado.

Los resultados obtenidos en este capítulo produjeron el siguiente artículo [1]:

- R. Acevedo, S. Meddahi, and R. Rodríguez, *An* $\boldsymbol{E}$*-based mixed formulation for a time-dependent eddy current problem*, Preprint 2008-03, Departamento de Ingeniería Matemática, Universidad de Concepción.

Este artículo fue sometido para publicación a la revista *Mathematics of Computation* y se encuentra en proceso de referato.

La formulación estudiada en este capítulo, encaja bien dentro de la teoría de operadores monótonos, debido a que la reluctividad (la inversa de la permeabilidad magnética) aparece como un coeficiente de difusión del problema parabólico correspondiente. En consecuencia, el modelo permite analizar el caso no lineal más típico, que aparece cuando se consideran materiales conductores ferromagnéticos. En el Cápitulo 4, analizamos la formulación mencionada en este tipo de materiales.

La existencia y unicidad de solución del problema (mixto, parabólico y no lineal) resultante, se estudia adaptando las técnicas del Capítulo 3 y usando la teoría de operadores monótonos. Las hipótesis impuestas sobre los parámetros físicos necesarias para el análisis, son perfectamente admisibles desde el punto de vista físico. Además, se consiguen estimaciones del error semejantes a las obtenidas en el capítulo 3. Sin embargo, si bien se obtienen estimaciones del error para la inducción magnética, la no linealidad impide extender la técnica usada en el Capítulo 3 para estimar el error correspondiente a las corrientes inducidas en el conductor.

En el Capítulo 5, se extiende la formulación analizada en el Capítulo 3 en un dominio acotado al problema considerado en todo el espacio. En contraste, con las condiciones homogéneas sobre la frontera (del dominio acotado) supuestas en el problema modelo estudiado en el Capítulo 3, en el Capítulo 5 se reducen las ecuaciones que satisfacen los campos en el dominio exterior (complemento del dominio acotado), a través de condiciones de contorno sobre la frontera de acoplamiento, obtenidas usando una representación integral adecuada de la solución en el dominio exterior. Utilizando operadores integrales que aparecen típicamente en problemas de electromagnetismo (ver, por ejemplo, [47]), se obtiene una formulación que permite combinar el método de los elementos finitos (en el dominio acotado) con el método de elementos de frontera (en la frontera de acoplamiento).

La existencia y unicidad de solución de la formulación FEM-BEM obtenida, se estudia eliminando la variable en la frontera, lo que permite reducir el problema a un problema mixto similar al estudiado en el Capítulo 3. En consecuencia, técnicas similares a las usadas en dicho capítulo permiten demostrar que el problema está bien planteado. Sin embargo, la no homogeneidad de la condición de contorno no permite demostrar que la seminorma de $\mathbf{H}(\mathbf{curl};\Omega)$ es una norma en el núcleo de la forma bilineal asociada a las restricciones (1.20)-(1.21), lo cual es imperativo en el análisis. Para superar esta dificultad, se optó por modificar el campo eléctrico en el material dieléctrico, de forma que se obtiene una condición de contorno homogénea alternativa, con la que también se genera el resultado deseado.

La selección del dominio computacional simplemente conexo con frontera conexa, permite usar elementos finitos usuales para aproximar la variable de la frontera. Así, el esquema semidiscreto se obtiene usando elementos de Nédélec para la variable principal y elementos finitos usuales, tanto para el multiplicador, como para la variable en la frontera de acoplamiento. Nuevamente la discretización temporal, en el esquema completamente discreto, se hace a través de un esquema implícito de Euler. En ambos casos, se deduce que los problemas correspondientes estan bien planteados, usando estrategias similares a las usadas en el Capítulo 3. De igual forma, pero teniendo cuidado con los términos en los que aparecen los operadores integrales, se deducen las estimaciones de error correspondientes, incluyendo las de las variables físicas que se obtuvieron en el Capítulo 3.

Dado que el conductor y la fuente de corriente estan contenidas en el interior del dominio computacional en cuya frontera se hace el acoplamiento, no hay dificultad en extender el análisis presentado en el Capítulo 5, incluyendo el caso no lineal analizado en el Capítulo 4. Como producto de la extensión de los resultados de los Capítulos 4 y 5, se encuentra en preparación el siguiente artículo:

- R. Acevedo, S. Meddahi, and R. Rodríguez, *An* $\boldsymbol{E}$*-based mixed-FEM and BEM coupling for a nonlinear magnetic field time-dependent eddy current problem.*

Chapter 2

Analysis of a potential formulation for the time-harmonic eddy current problem in a bounded domain

2.1 Introduction

The mathematical and numerical analysis of Maxwell equations has experimented an important development in different areas of applied mathematics and engineering during the last thirty years. We refer the reader to the books by Bossavit [27], Monk [57] and Silvester and Ferrari [67], as a representative sampling of text books devoted to numerical solution of electromagnetic problems.

Among the numerical methods found in the literature to approximate Maxwell equations, the finite element method is the most extended. See for instance [20] for a survey on this subject including a large list of references. Nowadays, it is the basis of several commercial codes such us Ansys, Femlab, Flux, Magnet, MSC/Emas, Opera, etc. We refer the reader to [68] for a description of most of these codes and further references.

The eddy current problem is obtained from Maxwell equations by assuming that all fields are harmonic and the frequency is low enough as to neglect the electric displacement in Ampère's Law. Such a situation happens, for instance, in problems related to electric machines working at power frequencies and in non-destructive materials testing.

In most practical situations, it is necessary to solve the electromagnetic problem in a bounded domain which contains conducting and non-conducting material (dielectrics), the equations in these two parts being typically of different kind. Moreover, the treatment of multiply connected conductors or dielectrics in three-dimensional domains present special difficulties. The choice of the unknowns in each subdomain is a crucial point to analyze the problem in domains with a general topology.

An important number of formulations and finite element methods to solve the eddy current problem in three-dimensional bounded domains can be found in the literature. There is a group of papers devoted to solve the problem in terms of certain scalar and vector potentials ([4, 5, 22, 23, 47, 58, 59]) and another group using formulations in terms of the magnetic field ([11, 8, 9, 16, 17, 19, 69]) or the electric field ([10, 18, 51]).

A thorough mathematical analysis of the formulations in terms of the magnetic or the electric field has been recently performed. This is not the case instead for formulations in terms of scalar and vector potentials. Indeed, in spite of the fact that the latter are the most frequently used in applications, there is only a very small number of papers dealing with their mathematical analysis. Among them, we mention a paper by Alonso *et al.* [7], where the well-posedness of some of these formulations is analyzed, and another one by Bíró & Valli [24] with the analysis of one such formulation in a general topological setting.

Different potentials have been used for the eddy current problem: a vector potential $\boldsymbol{A}$ for the magnetic field, a scalar potential V for the electric field in the conducting domain, a scalar potential ψ in dielectric domains, etc. A hierarchy of formulations involving these potentials have been discussed by Bíró & Preis [23]. In particular, they conclude that the so-called $\boldsymbol{A}, V - \boldsymbol{A} - \psi$ formulation, which involves all of them, is the most convenient in terms of computer cost. Numerical experiments illustrating the performance of this approach are also reported in this reference.

The aim of this paper is to provide a rigorous mathematical analysis of this formulation. Under rather general topological conditions, we prove that it leads to a well-posed problem, which can be numerically approximated by standard nodal finite elements. We also prove error estimates for the resulting numerical method. These estimates are valid as long as the three potentials are sufficiently smooth.

The smoothness of the scalar potentials V and ψ only relies on that of the original physical variables of the problem: the magnetic and the electric fields. Instead, the smoothness of the vector potential $\boldsymbol{A}$ also depends on the geometry of the domain chosen to define this non-physical variable. In principle this domain can be chosen freely, as far as it contains the conductors and the source currents. However, when it is chosen so that its connected components are either convex polyhedra or simply connected domains with smooth boundaries, the smoothness of $\boldsymbol{A}$ is mainly determined by the regularity of another physical variable: the magnetic induction field.

Because of this, we make such a choice for the domain of $\boldsymbol{A}$, which is not restrictive in practice. However, it is convenient to choose it as small as possible, because the magnetic field is written in terms of the more economical scalar potential ψ outside this domain. Thus, in the applications, the domain of $\boldsymbol{A}$ typically consists of a union of disjoint boxes, as small as possible, containing the current source and the conductors.

The outline of the paper is as follows: We introduce the eddy current problem and discuss the topological setting in Section 2.2. The $\boldsymbol{A}, V-\boldsymbol{A}-\psi$ potential formulation is introduced in Section 2.3. The corresponding variational problem is obtained in Section 2.4, where we also prove its well-posedness. Finally, in Section 2.5, we prove error estimates for a standard finite element method to solve numerically the problem. We also discuss in this section the convenience of choosing a domain with convex connected components for the vector potential. Throughout this chapter, c and C, with or without subscripts, bars, tildes or hats, denote positive constants, independent of the parameters and functions involved, which may take different values at different occurrences.

2.2 Eddy current problem

We consider a standard eddy current problem: to determine the electromagnetic fields induced in a three-dimensional conducting domain Ω_{C} by a given source current density $\boldsymbol{J}_{\mathrm{d}}$. We assume that the support of $\boldsymbol{J}_{\mathrm{d}}$ is compact and disjoint with Ω_{C}. The eddy current problem is in principle posed in the whole space. However, we restrict it to a bounded domain Ω containing both, Ω_{C} and the support of $\boldsymbol{J}_{\mathrm{d}}$, such that adequate boundary conditions can be imposed on its boundary. To this aim, we choose the geometry of Ω as simple as possible (v.g., simply connected with a connected boundary). See Fig. 2.1 for a two-dimensional sketch.

Let $\Omega_{\mathrm{C}} \subset \mathbb{R}^3$ be an open and bounded set with boundary Γ_{C}. Let $\Omega \subset \mathbb{R}^3$ be a simply connected bounded domain with a connected boundary Γ, such that $\overline{\Omega}_{\mathrm{C}} \subset \Omega$. We suppose that both, Ω and Ω_{C}, are either Lipschitz polyhedra or domains with $\mathcal{C}^{1,1}$ boundaries. We denote by $\boldsymbol{n}$ and $\boldsymbol{n}_{\mathrm{C}}$ the outward unit normal vectors to Ω and Ω_{C}, respectively, and by $\Omega_{\mathrm{D}} := \Omega \setminus \overline{\Omega}_{\mathrm{C}}$ the subdomain of Ω occupied by dielectric material, which includes the

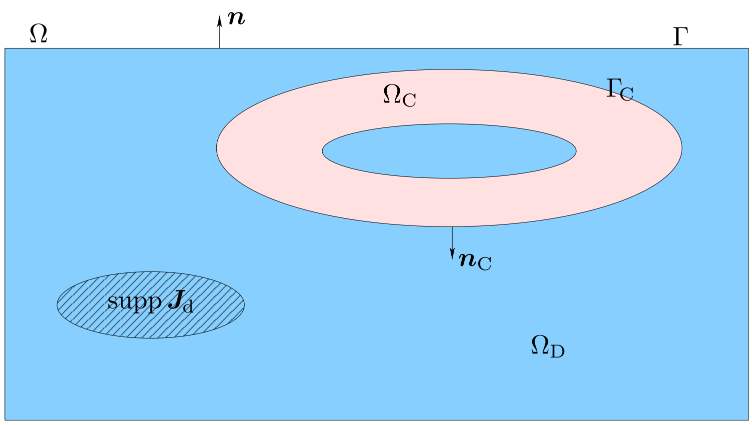

Figure 2.1: Two-dimensional sketch of the domain.

support of the source current (see Fig. 2.1). We will use standard notation for Sobolev spaces and norms.

The eddy current problem reads as follows:

Find $\boldsymbol{E} \in \mathbf{H}(\mathbf{curl}; \Omega_{\mathrm{C}})$ *and* $\boldsymbol{H} \in \mathbf{H}(\mathbf{curl}; \Omega)$ *such that:*

$$\mathbf{curl}\, \boldsymbol{H} = \sigma \boldsymbol{E} \qquad \text{in } \Omega_{\mathrm{C}}, \tag{2.1}$$

$$i\omega\mu \boldsymbol{H} + \mathbf{curl}\, \boldsymbol{E} = \mathbf{0} \qquad \text{in } \Omega_{\mathrm{C}}, \tag{2.2}$$

$$\mathbf{curl}\, \boldsymbol{H} = \boldsymbol{J}_{\mathrm{d}} \qquad \text{in } \Omega_{\mathrm{D}}, \tag{2.3}$$

$$\operatorname{div}(\mu \boldsymbol{H}) = 0 \qquad \text{in } \Omega, \tag{2.4}$$

$$\boldsymbol{H} \times \boldsymbol{n} = \boldsymbol{f}_{\mathrm{d}} \qquad \text{on } \Gamma. \tag{2.5}$$

The unknowns $\boldsymbol{E}$ and $\boldsymbol{H}$ are the magnetic and electric fields, respectively. The magnetic permeability μ and the conductivity σ are bounded functions satisfying:

$$0 < \mu_{\min} \leq \mu \leq \mu_{\max} \qquad \text{in } \Omega,$$

$$0 < \sigma_{\min} \leq \sigma \leq \sigma_{\max} \qquad \text{in } \Omega_{\mathrm{C}}.$$

Let us remark that the magnetic field has to satisfy the following coupling conditions:

$$\boldsymbol{H}|_{\Omega_{\mathrm{C}}} \times \boldsymbol{n}_{\mathrm{C}} = \boldsymbol{H}|_{\Omega_{\mathrm{D}}} \times \boldsymbol{n}_{\mathrm{C}} \qquad \text{on } \Gamma_{\mathrm{C}},$$

$$(\mu \boldsymbol{H})|_{\Omega_{\mathrm{C}}} \cdot \boldsymbol{n}_{\mathrm{C}} = (\mu \boldsymbol{H})|_{\Omega_{\mathrm{D}}} \cdot \boldsymbol{n}_{\mathrm{C}} \qquad \text{on } \Gamma_{\mathrm{C}}.$$

In fact, the latter is a consequence of (2.4), whereas the former follows from the fact that $\boldsymbol{H}$ must belong to $\mathbf{H}(\mathbf{curl}; \Omega)$.

The data of the problem are the source current density $\boldsymbol{J}_{\rm d} \in L^2(\Omega)^3$, for which we assume

$$\operatorname{supp} \boldsymbol{J}_{\rm d} \subset \Omega_{\rm D} \qquad \text{and} \qquad \operatorname{div} \boldsymbol{J}_{\rm d} = 0 \quad \text{in } \Omega_{\rm D},$$

and the tangential trace of the magnetic field $\boldsymbol{f}_{\rm d}$. Precise assumptions on $\boldsymbol{f}_{\rm d}$ will be made in Section 2.4 below; they essentially mean that $\boldsymbol{f}_{\rm d}$ has to be the tangential trace of a curl-free vector field (recall that $\mathbf{curl}\, \boldsymbol{H}$ vanishes in the neighborhood of Γ).

Equations (2.1)–(2.5) are enough to determine $\boldsymbol{E}$ and $\boldsymbol{H}$ only if the topology of the conducting domain $\Omega_{\rm C}$ is trivial. Otherwise, additional constraints must be imposed. To do this, we reduce our analysis to domains satisfying a standard topological assumption (see for instance Amrouche *et al.* [14]). We assume that there exists $m_{\rm D}$ connected open surfaces Σ_k (so called "cuts") contained in $\Omega_{\rm D}$, such that:

(i) each surface Σ_k is an open part of a smooth manifold,

(ii) the boundary of each Σ_k is contained in $\Gamma_{\rm C}$,

(iii) the intersection $\overline{\Sigma}_i \cap \overline{\Sigma}_j$ is empty for $i \neq j$,

(iv) the open set $\widehat{\Omega}_{\rm D} := \Omega_{\rm D} \setminus \bigcup_k \Sigma_k$ is pseudo-Lipschitz and simply connected.

Under this assumption, since Γ is connected, the space of *harmonic fields*

$$\begin{aligned} \mathscr{H}_\mu(\Gamma, \Gamma_{\rm C}) := \big\{ \boldsymbol{v} \in L^2(\Omega_{\rm D})^3 :\ & \mathbf{curl}\, \boldsymbol{v} = \mathbf{0} \text{ in } \Omega_{\rm D},\ \operatorname{div}(\mu \boldsymbol{v}) = 0 \text{ in } \Omega_{\rm D}, \\ & \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma \text{ and } \mu \boldsymbol{v} \cdot \boldsymbol{n}_{\rm C} = 0 \text{ on } \Gamma_{\rm C} \big\} \end{aligned}$$

satisfies $\dim \mathscr{H}_\mu(\Gamma, \Gamma_{\rm C}) = m_{\rm D}$ (see, for instance Fernandez & Gilardi [41, Proposition 5.6]). A basis for this space is given by $\{\mathbf{grad}\, \varphi_j\}_{j=1}^{m_{\rm D}}$, where each $\varphi_j \in H^1_\Gamma(\Omega_{\rm D} \backslash \Sigma_j)$ is the solution of the following elliptic problem:

$$\begin{aligned} [\![\varphi_j]\!]_{\Sigma_k} &= \delta_{jk}, \qquad && k = 1, \ldots, m_{\rm D}, \\ \int_{\Omega_{\rm D} \backslash \Sigma_j} \mu\, \mathbf{grad}\, \varphi_j \cdot \mathbf{grad}\, \chi &= 0 && \forall \chi \in H^1_\Gamma(\Omega_{\rm D}). \end{aligned}$$

In the expression above $[\![\cdot]\!]_{\Sigma_k}$ denotes the jump across Σ_k. Here and thereafter the subscript Γ in $H^1_\Gamma(\cdot)$ refers to function in $H^1(\cdot)$ with a vanishing trace on Γ.

Notice that although in principle $\mathbf{grad}\, \varphi_j \in L^2(\Omega_{\rm D} \setminus \Sigma_j)$, the last equation implies that $\mu\, \mathbf{grad}\, \varphi_j$ is a divergence-free function in the whole $\Omega_{\rm D}$ (not only in $\Omega_{\rm D} \setminus \Sigma_j$). Moreover, since the jump $[\![\varphi_j]\!]_{\Sigma_k}$ is constant, $\mathbf{grad}\,\varphi_j$ has also a vanishing **curl** in the whole $\Omega_{\rm D}$ (and not only in $\Omega_{\rm D} \setminus \Sigma_j$, again). Thus, $\mathbf{grad}\,\varphi_j \in \mathscr{H}_\mu(\Gamma, \Gamma_{\rm C})$.

To determine a unique solution of the eddy current problem (2.1)–(2.5), it is enough to add the following constraints (see Alonso *et al.* [7]):

$$\int_{\Omega_{\rm D}} i\omega\mu \boldsymbol{H} \cdot \mathbf{grad}\,\varphi_j + \int_{\Gamma_{\rm C}} (\boldsymbol{E} \times \boldsymbol{n}_{\rm C}) \cdot \mathbf{grad}\,\varphi_j = 0, \qquad j = 1, \ldots, m_{\rm D}. \tag{2.6}$$

Let us remark that the second integral above has a weak sense for $\boldsymbol{E} \in \mathbf{H}(\mathbf{curl}; \Omega_{\rm C})$ and $\mathbf{grad}\,\varphi_j \in \mathbf{H}(\mathbf{curl}; \Omega_{\rm D})$, as was shown by Buffa & Ciarlet [31, 32] for Lipschitz polyhedra and by Buffa *et al.* [34] for arbitrary Lipschitz domains (see Section 2.4 below for a precise definition).

2.3 The $\boldsymbol{A}, V - \boldsymbol{A} - \psi$ potential formulation

In this section we recall a classical formulation of the eddy current problem in terms of three potentials, $\boldsymbol{A}$, V and ψ, which was introduced by Leonard & Rodger [52]. We refer to Bíró & Preis [23] for a detailed discussion, which also includes numerical tests showing the efficiency of this approach.

First, we introduce a magnetic vector potential $\boldsymbol{A}$ defined in a subdomain Ω_A of Ω, which contains the conducting domain and the support of the source current. This subdomain does not need to be connected, but each of its connected components will be chosen either convex or simply connected with a smooth boundary. The reason for such a choice will be discussed in Section 2.5 below. On the other hand, for the sake of discretization, it is convenient to choose a polyhedral domain Ω_A; moreover, outside Ω_A, we will use a scalar potential, which will consequently require much less degrees of freedom for its discretization. Because of this, Ω_A will be chosen as small as possible, but with convex polyhedral connected components containing $\Omega_{\rm C}$ and $\operatorname{supp} \boldsymbol{J}_{\rm d}$ (see Fig. 2.2).

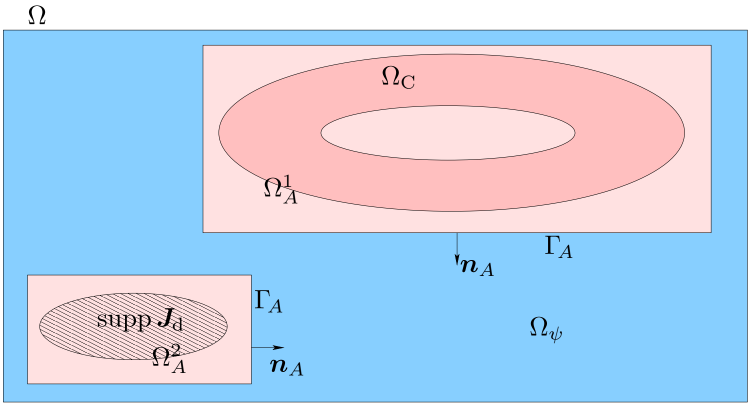

Figure 2.2: Two-dimensional sketch of the domains for the different potentials.

Let $\Omega_A \subset \mathbb{R}^3$ be an open set satisfying

$$\overline{\Omega}_{\mathrm{C}} \cup \operatorname{supp} \boldsymbol{J}_{\mathrm{d}} \subset \Omega_A \qquad \text{and} \qquad \overline{\Omega}_A \subset \Omega. \tag{2.7}$$

We denote by Ω_A^j, $j = 1, \ldots, m_A$, the connected components of Ω_A. We assume that each Ω_A^j is either a convex polyhedron or a simply connected domain with a $\mathcal{C}^{1,1}$ boundary, and that $\overline{\Omega}_A^j$ are mutually disjoint. We denote by Γ_A the boundary of Ω_A and by $\boldsymbol{n}_A$ its outward unit normal vector (see Fig. 2.2).

As a consequence of [43, Theorem I.3.5.], equation (2.4) implies that there exist unique $\boldsymbol{A}_j \in \mathbf{H}(\mathbf{curl}; \Omega_A^j)$ such that

$$\mu \boldsymbol{H} = \mathbf{curl}\, \boldsymbol{A}_j \qquad \text{in } \Omega_A^j, \tag{2.8}$$

$$\operatorname{div} \boldsymbol{A}_j = 0 \qquad \text{in } \Omega_A^j, \tag{2.9}$$

$$\boldsymbol{A}_j \cdot \boldsymbol{n}_A = 0 \qquad \text{on } \partial\Omega_A^j. \tag{2.10}$$

Thus, if we define $\boldsymbol{A} : \Omega_A \longrightarrow \mathbb{C}$ by

$$\boldsymbol{A}|_{\Omega_A^j} := \boldsymbol{A}_j, \qquad j = 1, \ldots, m_A,$$

then $\boldsymbol{A}$ belongs to the space

$$\boldsymbol{\mathcal{X}} := \mathbf{H}_0(\operatorname{div}; \Omega_A) \cap \mathbf{H}(\mathbf{curl}; \Omega_A),$$

whose natural norm is

$$\|\boldsymbol{z}\|_{\boldsymbol{\mathcal{X}}} := \left(\|\boldsymbol{z}\|_{0,\Omega_A}^2 + \|\operatorname{div} \boldsymbol{z}\|_{0,\Omega_A}^2 + \|\mathbf{curl}\, \boldsymbol{z}\|_{0,\Omega_A}^2 \right)^{\frac{1}{2}}.$$

Next, according to Bíró & Preis [23] (see also Bíró [22] and Bíró & Valli [24]) we introduce an electric scalar potential $V \in H^1(\Omega_{\mathrm{C}})$, such that

$$\boldsymbol{E} = -i\omega \boldsymbol{A} - i\omega\, \mathbf{grad}\, V \qquad \text{in } \Omega_{\mathrm{C}}. \tag{2.11}$$

Let us remark that (2.6) is a necessary condition for a global potential V to exist (see [7] and the formal argument at the end of this section). Notice that, from (2.1),

$$\operatorname{div}\left(-i\omega\sigma \boldsymbol{A} - i\omega\sigma\, \mathbf{grad}\, V\right) = 0 \qquad \text{in } \Omega_{\mathrm{C}}.$$

Moreover, since $\boldsymbol{H} \in \mathbf{H}(\mathbf{curl};\Omega)$, (2.1) and (2.3) also imply that

$$(i\omega\sigma \boldsymbol{A} + i\omega\sigma\, \mathbf{grad}\, V) \cdot \boldsymbol{n}_{\mathrm{C}} = 0 \qquad \text{on } \Gamma_{\mathrm{C}}.$$

These last two equations will be also collected in the potential formulation.

Equation (2.11) determines the electric potential V on each connected component of Ω_{C} up to an additive constant. Thus, if Ω_{C} has m_{C} connected components Ω_{C}^j, then the natural space for V is

$$\mathcal{M} := \prod_{j=1}^{m_{\mathrm{C}}} H^1(\Omega_{\mathrm{C}}^j)/\mathbb{C},$$

endowed with the norm $\|\mathbf{grad}\, V\|_{0,\Omega_{\mathrm{C}}}$.

Finally, a magnetic scalar potential ψ is defined in

$$\Omega_\psi := \Omega \setminus \overline{\Omega}_A$$

(see Fig. 2.2). To do this, notice that since Ω_A is a disjoint union of convex sets with $\overline{\Omega}_A \subset \Omega$ and Ω is simply connected, it turns out that Ω_ψ is simply connected too. Therefore, from (2.3) and (2.7) we know that there exists $\psi \in H^1(\Omega_\psi)$ (unique up to an additive constant) such that

$$\boldsymbol{H} = \omega\, \mathbf{grad}\, \psi \qquad \text{in } \Omega_\psi.$$

Thus, we are lead to the following formulation of problem (2.1)–(2.6) in terms of the potentials $\boldsymbol{A} \in \boldsymbol{\mathcal{X}}$, $V \in \mathcal{M}$ and $\psi \in H^1(\Omega_\psi)/\mathbb{C}$:

$$\mathbf{curl}\left(\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}\right) + i\omega\sigma\boldsymbol{A} + i\omega\sigma\,\mathbf{grad}\,V = \mathbf{0} \qquad \text{in } \Omega_{\mathrm{C}}, \tag{2.12}$$

$$\operatorname{div}(-i\omega\sigma\boldsymbol{A} - i\omega\sigma\,\mathbf{grad}\,V) = 0 \qquad \text{in } \Omega_{\mathrm{C}}, \tag{2.13}$$

$$\mathbf{curl}\left(\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}\right) = \boldsymbol{J}_{\mathrm{d}} \qquad \text{in } \Omega_A \setminus \overline{\Omega}_{\mathrm{C}}, \tag{2.14}$$

$$\left(\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}\right)\Big|_{\Omega_{\mathrm{C}}} \times \boldsymbol{n}_{\mathrm{C}} - \left(\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}\right)\Big|_{\Omega_A\setminus\overline{\Omega}_{\mathrm{C}}} \times \boldsymbol{n}_{\mathrm{C}} = \mathbf{0} \qquad \text{on } \Gamma_{\mathrm{C}}, \tag{2.15}$$

$$\operatorname{div}(\mu\,\mathbf{grad}\,\psi) = 0 \qquad \text{in } \Omega_\psi, \tag{2.16}$$

$$\operatorname{div}\boldsymbol{A} = 0 \qquad \text{in } \Omega_A, \tag{2.17}$$

$$\boldsymbol{A}\cdot\boldsymbol{n}_A = 0 \qquad \text{on } \Gamma_A, \tag{2.18}$$

$$\mathbf{grad}\,\psi \times \boldsymbol{n} = \boldsymbol{f}_{\mathrm{d}} \qquad \text{on } \Gamma, \tag{2.19}$$

$$\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}\cdot\boldsymbol{n}_A - \omega\,\mathbf{grad}\,\psi\cdot\boldsymbol{n}_A = 0 \qquad \text{on } \Gamma_A, \tag{2.20}$$

$$\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}\times\boldsymbol{n}_A - \omega\,\mathbf{grad}\,\psi\times\boldsymbol{n}_A = \mathbf{0} \qquad \text{on } \Gamma_A, \tag{2.21}$$

$$(i\omega\sigma\boldsymbol{A} + i\omega\sigma\,\mathbf{grad}\,V)\cdot\boldsymbol{n}_{\mathrm{C}} = 0 \qquad \text{on } \Gamma_{\mathrm{C}}. \tag{2.22}$$

Let us remark that (2.15) and (2.21) are consequences of the fact that $\boldsymbol{H} \in \mathbf{H}(\mathbf{curl};\Omega)$, whereas (2.20) follows from the fact that $\mu\boldsymbol{H} \in H(\operatorname{div};\Omega)$, which in its turn is a consequence of (2.4)

To end this section we show that any solution of the above equations yields a solution of the eddy current problem (2.1)–(2.6). In fact, let $(\boldsymbol{A}, V, \psi) \in \boldsymbol{\mathcal{X}} \times \mathcal{M} \times H^1(\Omega_\psi)/\mathbb{C}$ satisfying (2.12)–(2.22). Let

$$\boldsymbol{H} := \begin{cases} \dfrac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A} & \text{in } \Omega_A, \\ \omega\,\mathbf{grad}\,\psi & \text{in } \Omega_\psi, \end{cases} \tag{2.23}$$

and $\boldsymbol{E}$ be defined by (2.11). It is immediate to show that $\boldsymbol{H} \in \mathbf{H}(\mathbf{curl};\Omega)$, $\boldsymbol{E} \in \mathbf{H}(\mathbf{curl};\Omega_{\mathrm{C}})$, and they satisfy (2.1)–(2.5). There only remains to prove that (2.6) also holds true. To do this, notice that (2.4) implies that there exists a vector potential $\boldsymbol{B} \in \mathbf{H}(\mathbf{curl};\Omega)$ such that

$$\mu\boldsymbol{H} = \mathbf{curl}\,\boldsymbol{B} \qquad \text{in } \Omega. \tag{2.24}$$

Taking into account that the sets $\overline{\Omega}_A^j$ are simply connected and mutually disjoint, from

(2.23) there follows that there exists $\xi \in H^1(\Omega_A)$ such that

$$\boldsymbol{A} = \boldsymbol{B} + \mathbf{grad}\,\xi \qquad \text{in } \Omega_A.$$

Consequently, if we define $\widetilde{V} := V + \xi|_{\Omega_\mathrm{C}}$, we obtain from (2.11) that

$$\boldsymbol{E} = -i\omega(\boldsymbol{B} + \mathbf{grad}\,\widetilde{V}) \qquad \text{in } \Omega_\mathrm{C}. \tag{2.25}$$

Equations (2.24) and (2.25) fall in the framework analyzed by Alonso *et al.* [7, Section 6 (ii)], where it is shown that (2.6) holds true. This can be formally verified by using (2.23), the fact that $\mathbf{grad}\,\varphi_j \in \mathscr{H}_\mu(\Gamma, \Gamma_\mathrm{C})$ and (2.25), as follows:

$$\begin{aligned}
\int_{\Omega_\mathrm{D}} i\omega\mu\boldsymbol{H}\cdot\mathbf{grad}\,\varphi_j &= \int_{\Omega_\mathrm{D}} i\omega\,\mathbf{curl}\,\boldsymbol{B}\cdot\mathbf{grad}\,\varphi_j \\
&= i\omega\int_{\Gamma_\mathrm{C}} (\boldsymbol{B}\times\boldsymbol{n}_\mathrm{C})\cdot\mathbf{grad}\,\varphi_j \\
&= -\int_{\Gamma_\mathrm{C}} (\boldsymbol{E}\times\boldsymbol{n}_\mathrm{C})\cdot\mathbf{grad}\,\varphi_j - i\omega\int_{\Gamma_\mathrm{C}} (\mathbf{grad}\,\widetilde{V}\times\boldsymbol{n}_\mathrm{C})\cdot\mathbf{grad}\,\varphi_j \\
&= -\int_{\Gamma_\mathrm{C}} (\boldsymbol{E}\times\boldsymbol{n}_\mathrm{C})\cdot\mathbf{grad}\,\varphi_j,
\end{aligned}$$

where for the last equality we have used that

$$\begin{aligned}
\int_{\Gamma_\mathrm{C}} (\mathbf{grad}\,\widetilde{V}\times\boldsymbol{n}_\mathrm{C})\cdot\mathbf{grad}\,\varphi_j = &\int_{\Omega_\mathrm{D}} \mathbf{grad}\,\widetilde{V}^*\cdot\mathbf{curl}\,(\mathbf{grad}\,\varphi_j) \\
&- \int_{\Omega_\mathrm{D}} \mathbf{curl}(\mathbf{grad}\,\widetilde{V}^*)\cdot\mathbf{grad}\,\varphi_j = 0,
\end{aligned}$$

with $\widetilde{V}^* \in H^1(\Omega)$ being an extension of $\widetilde{V}$ to the whole Ω.

2.4 Variational formulation. Existence and uniqueness of solution

The aim of this section is to give a variational formulation of problem (2.12)–(2.22) and to prove its well-posedness.

First, we recall some results settled in [34] for Lipschitz domains. We write these results for Ω_A, as will be used in the sequel. The tangential trace operator $\gamma_\tau(\boldsymbol{u}) := \boldsymbol{u}|_{\Gamma_A}\times\boldsymbol{n}_A$ is a bounded linear operator from $\mathbf{H}(\mathbf{curl};\Omega_A)$ onto $H^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma_A)$. The tangential projection $\pi_\tau(\boldsymbol{v}) := \boldsymbol{n}_A\times(\boldsymbol{v}|_{\Gamma_A}\times\boldsymbol{n}_A)$ is a bounded linear operator from $\mathbf{H}(\mathbf{curl};\Omega_A)$ onto

$H^{-\frac{1}{2}}(\mathrm{curl}_\Gamma;\Gamma_A)$. Thus, the duality pairing between $H^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma_A)$ and $H^{-\frac{1}{2}}(\mathrm{curl}_\Gamma;\Gamma_A)$ is well defined by

$$\langle\gamma_\tau(\boldsymbol{u}),\pi_\tau(\boldsymbol{v})\rangle_{\Gamma_A} := \int_{\Omega_A}\mathbf{curl}\,\boldsymbol{u}\cdot\boldsymbol{v} - \int_{\Omega_A}\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v} \qquad \forall\boldsymbol{u},\boldsymbol{v}\in\mathbf{H}(\mathbf{curl};\Omega_A).$$

For any $\boldsymbol{w}\in\mathbf{H}(\mathbf{curl};\Omega_\psi)$, its tangential trace on Γ_A also belongs to $H^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma_A)$ and, consequently, $\langle\boldsymbol{w}\times\boldsymbol{n}_A,\pi_\tau(\boldsymbol{v})\rangle_{\Gamma_A}$ is also well defined.

To obtain a variational formulation of problem (2.12)–(2.22), notice that by virtue of (2.12), (2.14) and (2.15) we have that $\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}\in\mathbf{H}(\mathbf{curl};\Omega_A)$, and for all $\boldsymbol{z}\in\boldsymbol{\mathcal{X}}$

$$\int_{\Omega_A}\mathbf{curl}\left(\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}\right)\cdot\bar{\boldsymbol{Z}} = -i\omega\int_{\Omega_C}\sigma\left(\boldsymbol{A}+\mathbf{grad}\,V\right)\cdot\bar{\boldsymbol{Z}} + \int_{\Omega_A}\boldsymbol{J}_\mathrm{d}\cdot\bar{\boldsymbol{Z}}.$$

Integrating by parts the left-hand side above and using (2.17) and (2.21), there follows

$$\int_{\Omega_A}\frac{1}{\mu}\left[\mathbf{curl}\,\boldsymbol{A}\cdot\mathbf{curl}\,\bar{\boldsymbol{Z}} + (\mathrm{div}\,\boldsymbol{A})\left(\mathrm{div}\,\bar{\boldsymbol{Z}}\right)\right] + i\omega\int_{\Omega_C}\sigma\boldsymbol{A}\cdot\bar{\boldsymbol{Z}} \tag{2.26}$$
$$+i\omega\int_{\Omega_C}\sigma\,\mathbf{grad}\,V\cdot\bar{\boldsymbol{Z}} - \omega\,\langle\mathbf{grad}\,\psi\times\boldsymbol{n}_A,\pi_\tau(\boldsymbol{z})\rangle_{\Gamma_A} = \int_{\Omega_A}\boldsymbol{J}_\mathrm{d}\cdot\bar{\boldsymbol{Z}}.$$

On the other hand, from (2.13), by integrating by parts and using (2.22) we have for all $U\in H^1(\Omega_C)$

$$i\omega\int_{\Omega_C}\sigma\boldsymbol{A}\cdot\mathbf{grad}\,\bar{U} + i\omega\int_{\Omega_C}\sigma\,\mathbf{grad}\,V\cdot\mathbf{grad}\,\bar{U} = 0. \tag{2.27}$$

Finally, for any $\varphi\in H^1_\Gamma(\Omega_\psi)$, from (2.16), by integrating by parts and using (2.20), we obtain

$$\omega\int_{\Omega_\psi}\mu\,\mathbf{grad}\,\psi\cdot\mathbf{grad}\,\bar{\varphi} + \int_{\Gamma_A}\mathbf{curl}\,\boldsymbol{A}\cdot\boldsymbol{n}_A\,\bar{\varphi} = 0,$$

where the last integral must be understood as the duality pairing between $H^{-\frac{1}{2}}(\Gamma_A)$ and $H^{\frac{1}{2}}(\Gamma_A)$. Now, let $\varphi^*\in H^1(\Omega)$ be an extension of φ to the whole Ω. Hence,

$$\int_{\Gamma_A}\mathbf{curl}\,\boldsymbol{A}\cdot\boldsymbol{n}_A\,\bar{\varphi} = \int_{\Omega_A}\mathbf{curl}\,\boldsymbol{A}\cdot\mathbf{grad}\,\bar{\varphi}^* = \left\langle\mathbf{grad}\,\bar{\varphi}\times\boldsymbol{n}_A,\pi_\tau(\bar{\boldsymbol{A}})\right\rangle_{\Gamma_A}.$$

Therefore, we obtain

$$\omega\int_{\Omega_\psi}\mu\,\mathbf{grad}\,\psi\cdot\mathbf{grad}\,\bar{\varphi} + \left\langle\mathbf{grad}\,\bar{\varphi}\times\boldsymbol{n}_A,\pi_\tau(\bar{\boldsymbol{A}})\right\rangle_{\Gamma_A} = 0. \tag{2.28}$$

Equations (2.26)–(2.28) together with the essential condition (2.19), provide the following variational formulation of problem (2.12)–(2.22):

Find $\boldsymbol{A} \in \boldsymbol{\mathcal{X}}$, $V \in \mathcal{M}$ *and* $\psi \in H^1(\Omega_\psi)/\mathbb{C}$ *such that:*

$$\mathbf{grad}\,\psi \times \boldsymbol{n} = \boldsymbol{f}_{\mathrm{d}} \qquad \textit{in } H^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma), \tag{2.29}$$

$$\begin{aligned}\int_{\Omega_A} \frac{1}{\mu}\left[\mathbf{curl}\,\boldsymbol{A}\cdot\mathbf{curl}\,\bar{\boldsymbol{Z}} + (\mathrm{div}\,\boldsymbol{A})\left(\mathrm{div}\,\bar{\boldsymbol{Z}}\right)\right] + i\omega\int_{\Omega_{\mathrm{C}}} \sigma\boldsymbol{A}\cdot\bar{\boldsymbol{Z}} \\ + i\omega\int_{\Omega_{\mathrm{C}}} \sigma\,\mathbf{grad}\,V\cdot\bar{\boldsymbol{Z}} - \omega\left\langle \mathbf{grad}\,\psi\times\boldsymbol{n}_A, \pi_\tau(\boldsymbol{z})\right\rangle_{\Gamma_A} = \int_{\Omega_A} \boldsymbol{J}_{\mathrm{d}}\cdot\bar{\boldsymbol{Z}} \\ \forall \boldsymbol{z}\in\boldsymbol{\mathcal{X}},\end{aligned} \tag{2.30}$$

$$i\omega\int_{\Omega_{\mathrm{C}}} \sigma\boldsymbol{A}\cdot\mathbf{grad}\,\bar{U} + i\omega\int_{\Omega_{\mathrm{C}}} \sigma\,\mathbf{grad}\,V\cdot\mathbf{grad}\,\bar{U} = 0 \qquad \forall U\in\mathcal{M}, \tag{2.31}$$

$$\omega\int_{\Omega_\psi} \mu\,\mathbf{grad}\,\psi\cdot\mathbf{grad}\,\bar{\varphi} + \left\langle \mathbf{grad}\,\bar{\varphi}\times\boldsymbol{n}_A, \pi_\tau(\bar{\boldsymbol{A}})\right\rangle_{\Gamma_A} = 0 \qquad \forall\varphi\in H^1_\Gamma(\Omega_\psi). \tag{2.32}$$

Our next goal is to prove that this variational problem has a unique solution. For this purpose, first of all notice that (2.29) can be satisfied only if $\boldsymbol{f}_{\mathrm{d}}$ is the tangential trace on Γ of a gradient. Thus, this additional hypothesis turns out necessary for the problem to have a solution. So, we make the following assumption:

$$\exists\eta\in H^1(\Omega_\psi): \qquad \boldsymbol{f}_{\mathrm{d}} = \mathbf{grad}\,\eta\times\boldsymbol{n} \qquad \text{in } H^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma). \tag{2.33}$$

Now, let $\mathscr{A}$ be the bilinear form defined on $\boldsymbol{\mathcal{X}}\times\mathcal{M}\times H^1(\Omega_\psi)/\mathbb{C}$ by

$$\begin{aligned}&\mathscr{A}((\boldsymbol{A},V,\psi),(\boldsymbol{z},U,\varphi)) \\ &\quad := \int_{\Omega_A} \frac{1}{\mu}\left[\mathbf{curl}\,\boldsymbol{A}\cdot\mathbf{curl}\,\bar{\boldsymbol{Z}} + (\mathrm{div}\,\boldsymbol{A})\left(\mathrm{div}\,\bar{\boldsymbol{Z}}\right)\right] + \omega^2\int_{\Omega_\psi} \mu\,\mathbf{grad}\,\psi\cdot\mathbf{grad}\,\bar{\varphi} \\ &\quad\quad + i\omega\int_{\Omega_{\mathrm{C}}} \sigma\left(\boldsymbol{A}+\mathbf{grad}\,V\right)\cdot\left(\bar{\boldsymbol{Z}}+\mathbf{grad}\,\bar{U}\right) \\ &\quad\quad - \omega\left\langle \mathbf{grad}\,\psi\times\boldsymbol{n}_A, \pi_\tau(\boldsymbol{z})\right\rangle_{\Gamma_A} + \omega\left\langle \mathbf{grad}\,\bar{\varphi}\times\boldsymbol{n}_A, \pi_\tau(\bar{\boldsymbol{A}})\right\rangle_{\Gamma_A}.\end{aligned}$$

Clearly, (2.29)–(2.32) can be equivalently written as follows:

Find $(\boldsymbol{A},V,\psi)\in\boldsymbol{\mathcal{X}}\times\mathcal{M}\times H^1(\Omega_\psi)/\mathbb{C}$ *such that:*

$$\mathbf{grad}\,\psi \times \boldsymbol{n} = \boldsymbol{f}_{\mathrm{d}} \qquad \textit{in } H^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma), \tag{2.34}$$

$$\mathscr{A}((\boldsymbol{A},V,\psi),(\boldsymbol{z},U,\varphi)) = \int_{\Omega_A} \boldsymbol{J}_{\mathrm{d}}\cdot\bar{\boldsymbol{Z}} \qquad \forall(Z,U,\varphi)\in\boldsymbol{\mathcal{X}}\times\mathcal{M}\times H^1_\Gamma(\Omega_\psi). \tag{2.35}$$

Theorem 2.4.1 *Under assumption* (2.33)*, the variational problem* (2.34)–(2.35) *has a unique solution.*

Proof. It is enough to show that $\mathscr{A}$ is elliptic, since, in such a case, the theorem follows from Lax-Milgram's Lemma.

To prove the ellipticity, for $(\boldsymbol{z}, U, \varphi) \in \boldsymbol{\mathcal{X}} \times \mathcal{M} \times H^1(\Omega_\psi)/\mathbb{C}$ we write

$$\begin{aligned}\mathscr{A}((\boldsymbol{z}, U, \varphi), (\boldsymbol{z}, U, \varphi)) = &\int_{\Omega_A} \frac{1}{\mu} \left(|\mathbf{curl}\, \boldsymbol{z}|^2 + |\operatorname{div} \boldsymbol{z}|^2\right) + \omega^2 \int_{\Omega_\psi} \mu \left|\mathbf{grad}\, \varphi\right|^2 \\ &+ i\omega \left\{ \int_{\Omega_C} \sigma \left(|\boldsymbol{z}|^2 + |\mathbf{grad}\, U|^2\right) + 2 \int_{\Omega_C} \sigma \operatorname{Re}(\mathbf{grad}\, U \cdot \bar{\boldsymbol{Z}}) \right. \\ &\left. + 2 \operatorname{Im} \left\langle \mathbf{grad}\, \bar{\varphi} \times \boldsymbol{n}_A, \pi_\tau(\bar{\boldsymbol{Z}}) \right\rangle_{\Gamma_A} \right\}.\end{aligned}$$

Thus,

$$|\mathscr{A}((\boldsymbol{z}, U, \varphi), (\boldsymbol{z}, U, \varphi))|^2 = (a + \omega^2 b)^2 + \omega^2 (c + 2d)^2,$$

where

$$a := \int_{\Omega_A} \frac{1}{\mu} \left(|\mathbf{curl}\, \boldsymbol{z}|^2 + |\operatorname{div} \boldsymbol{z}|^2\right), \qquad b := \int_{\Omega_\psi} \mu \left|\mathbf{grad}\, \varphi\right|^2,$$

$$c := \int_{\Omega_C} \sigma \left(|\boldsymbol{z}|^2 + |\mathbf{grad}\, U|^2\right), \qquad d := e + f,$$

with

$$e := \int_{\Omega_C} \sigma \operatorname{Re}(\mathbf{grad}\, U \cdot \bar{\boldsymbol{Z}}) \qquad \text{and} \qquad f := \operatorname{Im} \left\langle \mathbf{grad}\, \bar{\varphi} \times \boldsymbol{n}_A, \pi_\tau(\bar{\boldsymbol{Z}}) \right\rangle_{\Gamma_A}.$$

Next, we proceed as in [24] and use the elementary inequality

$$(c + 2d)^2 \geq \rho c^2 - 8\rho d^2 \qquad \forall c, d \in \mathbb{R},\ \forall \rho \in (0, 1/2],$$

to obtain

$$|\mathscr{A}((\boldsymbol{z}, U, \varphi), (\boldsymbol{z}, U, \varphi))|^2 \geq a^2 + \omega^4 b^2 + \omega^2 (\rho c^2 - 8\rho d^2) \qquad \forall \rho \in (0, 1/2].$$

Now, since[1]

$$a \geq \frac{K}{\mu_{\max}} \|\boldsymbol{z}\|^2_{\boldsymbol{\mathcal{X}}} \qquad \text{and} \qquad b \geq \mu_{\min} \|\mathbf{grad}\, \varphi\|^2_{0,\Omega_\psi},$$

[1] For the first inequality see for instance, Lemma I.3.6 from [43].

with $K > 0$ independent of $\boldsymbol{z}$, we have

$$|\mathscr{A}((\boldsymbol{z},U,\varphi),(\boldsymbol{z},U,\varphi))|^2 \geq \frac{K^2}{\mu_{\max}^2}\|\boldsymbol{z}\|_{\boldsymbol{\mathcal{X}}}^4 + \omega^4\mu_{\min}^2\|\mathbf{grad}\,\varphi\|_{0,\Omega_\psi}^4 + \omega^2\rho\left(\int_{\Omega_C}\sigma\,|\mathbf{grad}\,U|^2\right)^2 - 16\omega^2\rho(e^2+f^2).$$

To estimate the last term in the right-hand side above, notice first that, for all $\varepsilon > 0$,

$$e^2 \leq \left(\int_{\Omega_C}\left|\sigma\,\mathbf{grad}\,U\cdot\bar{\boldsymbol{Z}}\right|\right)^2 \leq \frac{\varepsilon}{2}\left(\int_{\Omega_C}\sigma\,|\mathbf{grad}\,U|^2\right)^2 + \frac{1}{2\varepsilon}\left(\int_{\Omega_C}\sigma\,|\boldsymbol{z}|^2\right)^2.$$

On the other hand, $\exists C > 0$ independent of φ and $\boldsymbol{z}$ such that

$$f^2 \leq \|\mathbf{grad}\,\bar{\varphi}\times\boldsymbol{n}_A\|_{H^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma_A)}^2\left\|\pi_\tau(\bar{\boldsymbol{Z}})\right\|_{H^{-\frac{1}{2}}(\mathrm{curl}_\Gamma;\Gamma_A)}^2 \leq C\left(\|\mathbf{grad}\,\varphi\|_{0,\Omega_\psi}^4 + \|\boldsymbol{z}\|_{\boldsymbol{\mathcal{X}}}^4\right).$$

Therefore, by combining the last three inequalities and taking ε and ρ small enough, we obtain that $\exists\alpha > 0$ such that, $\forall(\boldsymbol{z},U,\varphi)\in\boldsymbol{\mathcal{X}}\times\mathcal{M}\times H^1(\Omega_\psi)/\mathbb{C}$,

$$|\mathscr{A}((\boldsymbol{z},U,\varphi),(\boldsymbol{z},U,\varphi))|^2 \geq \alpha\left(\|\boldsymbol{z}\|_{\boldsymbol{\mathcal{X}}}^4 + \|\mathbf{grad}\,U\|_{0,\Omega_C}^4 + \|\mathbf{grad}\,\varphi\|_{0,\psi}^4\right),$$

which allows us to conclude the ellipticity of $\mathscr{A}$. □

To end this section, we prove that the unique solution of the variational problem (2.34)–(2.35) is actually a solution of the strong form of the problem given by equations (2.12)–(2.22).

Theorem 2.4.2 *The solution* $(\boldsymbol{A},V,\psi)$ *of* (2.34)–(2.35) *satisfies* (2.12)–(2.22).

Proof. First, let $\xi\in H^1(\Omega_A)$ be a solution of the compatible Neumann problem $\Delta\xi = \mathrm{div}\,\boldsymbol{A}$ in Ω_A, $\partial\xi/\partial\boldsymbol{n}_A = 0$ on Γ_A. By testing (2.30) with $\boldsymbol{z} = \mathbf{grad}\,\xi\in\boldsymbol{\mathcal{X}}$, we obtain (2.17) by using (2.31) (since $\xi|_{\Omega_C}\in\mathcal{M}$) and $\langle\mathbf{grad}\,\psi\times\boldsymbol{n}_A,\pi_\tau(\mathbf{grad}\,\xi)\rangle_{\Gamma_A} = 0$ (which is a consequence of the definition of the duality pairing).

Second, by testing (2.30)–(2.32) with smooth functions supported in adequate domains and proceeding in the standard way, it is easy to verify equations (2.12)–(2.16), (2.20) and (2.22). Since (2.18) is imposed in the definition of the space $\boldsymbol{\mathcal{X}}$ and (2.19) coincides with (2.34), there only remains to prove (2.21) in $H^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma_A)$; namely, that for all $\boldsymbol{\zeta}\in\mathbf{H}(\mathbf{curl};\Omega_A)$,

$$\left\langle\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}\times\boldsymbol{n}_A,\pi_\tau(\boldsymbol{\zeta})\right\rangle_{\Gamma_A} - \langle\mathbf{grad}\,\psi\times\boldsymbol{n}_A,\pi_\tau(\boldsymbol{\zeta})\rangle_{\Gamma_A} = 0. \tag{2.36}$$

To do this, notice first that by substituting (2.17) in (2.30), integrating by parts and having into account (2.12) and (2.14), we obtain

$$\left\langle \frac{1}{\mu} \mathbf{curl}\, \boldsymbol{A} \times \boldsymbol{n}_A, \pi_\tau(\boldsymbol{z}) \right\rangle_{\Gamma_A} - \langle \mathbf{grad}\, \psi \times \boldsymbol{n}_A, \pi_\tau(\boldsymbol{z}) \rangle_{\Gamma_A} = 0 \qquad \forall \boldsymbol{z} \in \boldsymbol{\mathcal{X}}.$$

Next, for $\boldsymbol{\zeta} \in \mathbf{H}(\mathbf{curl}; \Omega_A)$, let φ be a solution of the following auxiliary problem:

$$\varphi \in H^1(\Omega_A)/\mathbb{C} : \qquad \int_{\Omega_A} \mathbf{grad}\, \varphi \cdot \mathbf{grad}\, \bar{\chi} = \int_{\Omega_A} \boldsymbol{\zeta} \cdot \mathbf{grad}\, \bar{\chi} \qquad \forall \chi \in H^1(\Omega_A)/\mathbb{C}.$$

Hence, $\operatorname{div}(\boldsymbol{\zeta} - \mathbf{grad}\, \varphi) = 0$ in Ω_A and $(\boldsymbol{\zeta} - \mathbf{grad}\, \varphi) \cdot \boldsymbol{n}_A = 0$ on Γ_A. Consequently, $\boldsymbol{z} := \boldsymbol{\zeta} - \mathbf{grad}\, \varphi \in \boldsymbol{\mathcal{X}}$, and using it as a test function in the equation above, we obtain

$$\left\langle \frac{1}{\mu} \mathbf{curl}\, \boldsymbol{A} \times \boldsymbol{n}_A, \pi_\tau(\boldsymbol{\zeta} - \mathbf{grad}\, \varphi) \right\rangle_{\Gamma_A} - \langle \mathbf{grad}\, \psi \times \boldsymbol{n}_A, \pi_\tau(\boldsymbol{\zeta} - \mathbf{grad}\, \varphi) \rangle_{\Gamma_A} = 0.$$

Now, from (2.12) and (2.14), we have

$$\begin{aligned} \left\langle \frac{1}{\mu} \mathbf{curl}\, \boldsymbol{A} \times \boldsymbol{n}_A, \pi_\tau(\mathbf{grad}\, \varphi) \right\rangle_{\Gamma_A} &= \int_{\Omega_A} \mathbf{curl} \left(\frac{1}{\mu} \mathbf{curl}\, \boldsymbol{A} \right) \cdot \mathbf{grad}\, \bar{\varphi} \\ &= - \int_{\Omega_C} (i\omega\sigma \boldsymbol{A} + i\omega\sigma\, \mathbf{grad}\, V) \cdot \mathbf{grad}\, \bar{\varphi} \\ &\quad + \int_{\Omega_A} \boldsymbol{J}_{\mathrm{d}} \cdot \mathbf{grad}\, \bar{\varphi} \\ &= 0, \end{aligned}$$

where, for the last step, we have used integration by parts, (2.13), (2.22), the assumption that $\boldsymbol{J}_{\mathrm{d}}$ is divergence-free and (2.7).

Thus, using again that $\langle \mathbf{grad}\, \psi \times \boldsymbol{n}_A, \pi_\tau(\mathbf{grad}\, \varphi) \rangle_{\Gamma_A}$ vanishes, (2.36) follows from the last two equations, and we conclude the proof. □

2.5 Numerical approximation

In this section we describe and analyze a finite element method to approximate the solution of problem (2.34)–(2.35). To do this, first notice that (2.34) implies that the surface gradient of ψ can be written as follows:

$$\nabla_\Gamma \psi := \boldsymbol{n} \times (\nabla \psi \times \boldsymbol{n}) = \boldsymbol{n} \times \boldsymbol{f}_{\mathrm{d}}.$$

Therefore, if we take an arbitrary but fixed point $\boldsymbol{x}_0 \in \Gamma$ and if the data $\boldsymbol{f}_{\mathrm{d}}$ is sufficiently smooth (for instance, it is enough that $\boldsymbol{f}_{\mathrm{d}} \in H^{\frac{1}{2}+\delta}(\Gamma)^3$ with $\delta > 0$), then we can compute in advance the values of ψ on Γ as follows:

$$\psi(\boldsymbol{x}) = \int_{\boldsymbol{\alpha}(\boldsymbol{x})} \nabla_\Gamma \psi \cdot \boldsymbol{t}_{\boldsymbol{\alpha}(\boldsymbol{x})} = \int_{\boldsymbol{\alpha}(\boldsymbol{x})} \boldsymbol{n} \times \boldsymbol{f}_{\mathrm{d}} \cdot \boldsymbol{t}_{\boldsymbol{\alpha}(\boldsymbol{x})},$$

where $\boldsymbol{\alpha}(\boldsymbol{x})$ is any simple curve lying on Γ and joining $\boldsymbol{x}_0$ with $\boldsymbol{x}$, and $\boldsymbol{t}_{\boldsymbol{\alpha}(\boldsymbol{x})}$ is its unit tangent vector. Notice that the computed value of $\psi(\boldsymbol{x})$ is independent of the particular curve $\boldsymbol{\alpha}(\boldsymbol{x})$. Thus, if we define

$$g_{\mathrm{d}}(\boldsymbol{x}) := \int_{\boldsymbol{\alpha}(\boldsymbol{x})} \boldsymbol{n} \times \boldsymbol{f}_{\mathrm{d}} \cdot \boldsymbol{t}_{\boldsymbol{\alpha}(\boldsymbol{x})}, \tag{2.37}$$

then problem (2.34)–(2.35) is equivalent to the following one:

Find $(\boldsymbol{A}, V, \psi) \in \boldsymbol{\mathcal{X}} \times \mathcal{M} \times H^1(\Omega_\psi)$ *such that:*

$$\psi = g_{\mathrm{d}} \qquad \text{on } \Gamma, \tag{2.38}$$

$$\mathscr{A}((\boldsymbol{A}, V, \psi), (\boldsymbol{z}, U, \varphi)) = \int_{\Omega_A} \boldsymbol{J}_{\mathrm{d}} \cdot \bar{\boldsymbol{Z}} \qquad \forall (Z, U, \varphi) \in \boldsymbol{\mathcal{X}} \times \mathcal{M} \times H^1_\Gamma(\Omega_\psi). \tag{2.39}$$

To obtain a discrete formulation of this problem, we further assume that all the domains are Lipschitz polyhedra. Let $\{\mathcal{T}_h\}$ be a family of tetrahedral meshes of Ω such that, for each mesh, all the elements $T \in \mathcal{T}_h$ are completely included in one of the three subdomains $\overline{\Omega}_A$, $\overline{\Omega}_{\mathrm{C}}$ or $\overline{\Omega}_\psi$.

Consider the following finite element spaces:

$$\begin{aligned}
\boldsymbol{\mathcal{X}}_h &:= \left\{ \boldsymbol{z}_h \in \boldsymbol{\mathcal{X}} : \ \boldsymbol{z}_h|_T \in \mathbb{P}_m^3 \ \forall T \in \mathcal{T}_h : T \subset \overline{\Omega}_A \right\}, \\
\mathcal{M}_h &:= \left\{ U_h \in \mathcal{M} : \ U_h|_T \in \mathbb{P}_m \ \forall T \in \mathcal{T}_h : T \subset \overline{\Omega}_{\mathrm{C}} \right\}, \\
\mathcal{Q}_h &:= \left\{ \varphi_h \in H^1(\Omega_\psi) : \ \varphi_h|_T \in \mathbb{P}_m \ \forall T \in \mathcal{T}_h : T \subset \overline{\Omega}_\psi \right\}, \\
\mathcal{Q}_{\Gamma,h} &:= \left\{ \varphi_h \in \mathcal{Q}_h : \ \varphi_h|_\Gamma = 0 \right\},
\end{aligned}$$

where $\mathbb{P}_m$, $m \geq 1$, is the set of polynomials of degree not greater than m.

For the boundary condition, we choose the following discrete approximation of g_{d}:

$$g_{\mathrm{d},h} := \Pi_h^\Gamma g_{\mathrm{d}}, \tag{2.40}$$

where Π_h^Γ is the Lagrange interpolant on the triangular mesh on Γ which consists of the faces of tetrahedra of $T \in \mathcal{T}_h$ lying on Γ, that we denote $\mathcal{T}_h^\Gamma$. Notice that the definition of

$g_{\mathrm{d},h}$ makes sense because g_{d}, as defined by (2.37), is continuous. Let us remark that $g_{\mathrm{d},h}$ is completely determined by its values at the vertices of the triangulation $\mathcal{T}_h^{\Gamma}$, which can be conveniently computed from the data $\boldsymbol{f}_{\mathrm{d}}$ by means of (2.37), with $\boldsymbol{\alpha}(\boldsymbol{x})$ being a curve formed by edges of $\mathcal{T}_h^{\Gamma}$.

Thus, we are lead to the following discrete problem:

Find $(\boldsymbol{A}_h, V_h, \psi_h) \in \boldsymbol{\mathcal{X}}_h \times \mathcal{M}_h \times \mathcal{Q}_h$ *such that:*

$$\psi_h = g_{\mathrm{d},h} \qquad \text{on } \Gamma, \tag{2.41}$$

$$\mathscr{A}((\boldsymbol{A}_h, V_h, \psi_h), (\boldsymbol{z}_h, U_h, \varphi_h)) = \int_{\Omega_A} \boldsymbol{J}_{\mathrm{d}} \cdot \bar{\boldsymbol{Z}}_h \qquad \forall (\boldsymbol{z}_h, U_h, \varphi_h) \in \boldsymbol{\mathcal{X}}_h \times \mathcal{M}_h \times \mathcal{Q}_{\Gamma,h}. \tag{2.42}$$

The existence and uniqueness of the solution of this discrete problem is again an immediate consequence of the ellipticity of $\mathscr{A}$, proved in the proof of Theorem 2.4.2, and Lax-Milgram's Lemma. Moreover, if the solution of the continuous problem is smooth enough, the standard finite element error analysis techniques yield the following result:

Theorem 2.5.1 *Let* $g_{\mathrm{d}} \in \mathcal{C}(\Gamma)$ *and* $g_{\mathrm{d},h}$ *be defined by* (2.40)*. Let*

$$(\boldsymbol{A}, V, \psi) \text{ and } (\boldsymbol{A}_h, V_h, \psi_h)$$

be the solutions of problems (2.38)–(2.39) *and* (2.41)–(2.42)*, respectively.*

If $\boldsymbol{A} \in H^{1+s}(\Omega_A)^3$, $V \in H^{1+s}(\Omega_{\mathrm{C}})$ *and* $\psi \in H^{1+s}(\Omega_\psi)$ *with* $s > 0$*, then there exists a strictly positive constant* C*, independent of* h*,* $\boldsymbol{A}$*,* V *and* ψ*, such that*

$$\|\boldsymbol{A} - \boldsymbol{A}_h\|_{\boldsymbol{\mathcal{X}}} + \|\mathbf{grad}\,(V - V_h)\|_{0,\Omega_{\mathrm{C}}} + \|\mathbf{grad}\,(\psi - \psi_h)\|_{0,\Omega_\psi} \leq Ch^r \left(\|\boldsymbol{A}\|_{1+s,\Omega_A} + \|V\|_{1+s,\Omega_{\mathrm{C}}} + \|\psi\|_{1+s,\Omega_\psi} \right),$$

with $r := \min\{m, s\}$.

Proof. Let Π_h be the Lagrange interpolant on Q_h. Since

$$(\Pi_h \psi)|_\Gamma = \Pi_h^\Gamma \psi = \Pi_h^\Gamma g_{\mathrm{d}} = g_{\mathrm{d},h} = \psi_h|_\Gamma,$$

we have that $\psi_h - \Pi_h \psi \in Q_{\Gamma,h}$. Therefore, the theorem is a direct consequence of the ellipticity of $\mathscr{A}$, Cea's lemma and the approximation properties of the Lagrange interpolant (see, for instance, Ciarlet [35]). □

To end the paper we discuss the convenience of choosing the domain Ω_A of the vector potential so that its connected components be convex polyhedra. For simplicity, we take Ω_A connected in what follows, but all the statements hold true for each of its connected components. So let Ω_A be simply connected with a connected boundary.

According to [43, Theorem I.3.4], since $\operatorname{div}(\mu \boldsymbol{H}) = 0$ in Ω, there exists $\boldsymbol{\Phi} \in H^1(\Omega)^3$ satisfying:

$$\begin{aligned} \mathbf{curl}\, \boldsymbol{\Phi} &= \mu \boldsymbol{H} && \text{in } \Omega, \\ \operatorname{div} \boldsymbol{\Phi} &= 0 && \text{in } \Omega. \end{aligned}$$

Moreover, according to Remark I.3.12 of the same reference, if $\mu \boldsymbol{H} \in H^p(\Omega)^3$ with $0 < p \leq 1$, then $\boldsymbol{\Phi} \in H^{1+p}(\Omega)^3$.

Therefore, by virtue of (2.8)–(2.10), there holds:

$$\begin{aligned} \mathbf{curl}(\boldsymbol{A} - \boldsymbol{\Phi}) &= \mathbf{0} && \text{in } \Omega_A, \\ \operatorname{div}(\boldsymbol{A} - \boldsymbol{\Phi}) &= 0 && \text{in } \Omega_A, \\ (\boldsymbol{A} - \boldsymbol{\Phi}) \cdot \boldsymbol{n}_A &= -\boldsymbol{\Phi} \cdot \boldsymbol{n}_A && \text{on } \Gamma_A. \end{aligned}$$

The first equation above and the simple-connectedness of Ω_A implies that there exists a unique $\chi \in H^1(\Omega_A)/\mathbb{C}$ such that $\boldsymbol{A} - \boldsymbol{\Phi} = \mathbf{grad}\, \chi$ in Ω_A, whereas the remaining equations imply that χ is the solution of the following compatible Neumann problem:

$$\begin{aligned} \Delta \chi &= 0 && \text{in } \Omega_A, \\ \frac{\partial \chi}{\partial \boldsymbol{n}_A} &= -\boldsymbol{\Phi} \cdot \boldsymbol{n}_A && \text{on } \Gamma_A. \end{aligned}$$

The Neumann data of this problem will be in general smooth on each polygonal face F of Γ_A, since Γ_A is an arbitrary polyhedral surface within the dielectric domain. In fact, if $\mu \boldsymbol{H} \in H^p(\Omega)^3$ with $0 < p \leq 1$, then $\boldsymbol{\Phi}|_F \cdot \boldsymbol{n}_A \in H^{\frac{1}{2}+p}(F)$ for all faces F.

Therefore, if Ω_A is a convex polyhedron, then there exists $q > 0$ such that $\chi \in H^{2+q}(\Omega_A)$ (see [40]). Consequently,

$$\boldsymbol{A} = \boldsymbol{\Phi} + \mathbf{grad}\, \chi \in H^{1+s}(\Omega_A)^3,$$

with $s := \min\{p, q\} > 0$. Conversely, if Ω_A were a non-convex polyhedron, then, in general, $\chi \notin H^2(\Omega_A)$ and, consequently,

$$\boldsymbol{A} = \boldsymbol{\Phi} + \mathbf{grad}\, \chi \notin H^1(\Omega_A)^3.$$

In such a case, Theorem 2.5.1 would become meaningless.

Moreover, $\boldsymbol{\mathcal{Y}} := \{\boldsymbol{z} \in H^1(\Omega_A)^3 : \ \boldsymbol{z} \cdot \boldsymbol{n}_A = 0 \text{ on } \Gamma_A\}$ is a closed subspace of $\boldsymbol{\mathcal{X}}$ (see [38]). When Ω_A is a polyhedron, it is well-known that $\boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{X}}$ if and only if Ω_A is convex (see [43, Theorem I.3.9] and [38]).

The finite element space $\boldsymbol{\mathcal{X}}_h$ is clearly a subspace of $\boldsymbol{\mathcal{Y}}$. Therefore, when Ω_A is a convex polyhedron, it makes sense to approximate $\boldsymbol{A} \in \boldsymbol{\mathcal{X}}$ by finite elements from $\boldsymbol{\mathcal{X}}_h$.

Instead, if Ω_A were not convex, then there would be no hope of approximating $\boldsymbol{A}$ by finite elements from $\boldsymbol{\mathcal{X}}_h$. Indeed, as stated above, in general $\boldsymbol{A} \notin H^1(\Omega_A)^3$ in such a case. Hence, $\boldsymbol{A}$ would not belong to the closed set $\boldsymbol{\mathcal{Y}}$ containing the finite element spaces $\boldsymbol{\mathcal{X}}_h$ for all meshes. So, there could not exist $\boldsymbol{A}_h$ such that $\|\boldsymbol{A} - \boldsymbol{A}_h\|_{\boldsymbol{\mathcal{X}}} \to 0$ as h goes to zero.

2.6 Conclusions

We have proved that the $\boldsymbol{A}, V - \boldsymbol{A} - \psi$ formulation of the eddy current problem is well posed and that its discretization by standard nodal finite elements leads to an optimal-order numerical approximation. This gives mathematical support to the well-known efficiency of this approach in applications.

However, for the convergence of the numerical method, the connected components of the domain of the vector potential $\boldsymbol{A}$ must be chosen as convex polyhedra. Since this domain can be chosen freely (as far as it contains the conductors and the source current), this is not a severe restriction in practice.

Chapter 3

An E-based mixed formulation for a time-dependent eddy current problem

3.1 Introduction

Numerical solution of Maxwell equations is now an increasingly important research area in science and engineering. We refer the reader to the books by Bossavit [27], Monk [57], and Silvester and Ferrari [67], as a representative sampling of text books devoted to numerical solution of electromagnetic problems. Among the numerical methods found in the literature to approximate Maxwell equations, the finite element method is the most extended.

In applications related to electrical power engineering (see for instance [63]) the displacement current existing in a metallic conductor is negligible compared with the conduction current. In such situations the displacement currents can be dropped from Maxwell equations to obtain a magneto-quasistatic submodel usually called *eddy current problem*; see for instance [27, Chapter 8]. From the mathematical point of view, this submodel provides a reasonable approximation to the solution of the full Maxwell system in the low frequency range [13].

When dealing with alternating currents, the imposed current density shows a harmonic dependence on time. In such a case, the steady state electric and magnetic fields also have this harmonic behavior, leading to the so-called *time-harmonic* eddy current problem.

However, even in the case of a sinusoidal supply voltage, in some occasions one may need to simulate transient states. Besides, in some cases it is not possible to assume a sinusoidal behavior for the whole electromagnetic system. Actually, the present paper is intended as a first (linear) step towards the nonlinear case that happens in the presence of ferromagnetic materials. In this approach, we allow the magnetic permeability to be time-dependent and write the problem in terms of the electric field $\boldsymbol{E}$. In contrast to the $\boldsymbol{H}$-based formulation given in [56], the $\boldsymbol{E}$-formulation fits well into the theory of monotone operators, because the reluctivity (the inverse of the magnetic permeability) appears as a diffusion coefficient in the degenerate parabolic problem at hand (see (3.12) below).

Generally, the eddy current problem is posed in the whole space with decay conditions on the fields at infinity. There exist many techniques to tackle with this difficulty, for example a BEM-FEM strategy is used in [46, 55] in the harmonic regime case and in [56] in the transient case. Here we opted for a simpler approach: we restrict the equations to a sufficiently large domain Ω containing the region of interest and impose a convenient artificial boundary condition on its border. Although thorough mathematical and numerical analysis of several finite element formulations of the time-harmonic eddy current model in a bounded domain have been performed (see for instance Bermúdez *et al.* [16] and Alonso *et al.* [8]), this is not the case for the time-dependent problem.

The aim of this work is to propose a new formulation for the time-dependent eddy current model posed in a bounded domain, with no restrictions on the topology of the conductor or on the regularity of its boundary. This formulation is obtained by introducing a time primitive of $\boldsymbol{E}$ as the primary unknown and using a Lagrange multiplier associated to the divergence-free constraint satisfied by this variable in the insulating region surrounding the conductor. The techniques used to show that this saddle-point formulation is well-posed are similar the ones given in [21, 56]. (Other formulation for a time-dependent eddy current problem in terms of a magnetic vector potential is given in [15].) Mixed finite element schemes have been used extensively for the approximation of evolution problems, mainly in fluid dynamics applications; see for instance, Johnson and Thomée [49] and Bernardi and Raugel [21]. More recently, Boffi and Gastaldi [25] give sufficient conditions for the convergence of approximation for two types of mixed parabolic problems, the heat equation in mixed form being a model for the first case, while the time dependent Stokes problem fits into the second one.

We perform a space discretization of our weak formulation by using Nédélec edge elements (see [61]) for the main variable and standard finite elements for the Lagrange multiplier. We show that our semi-discrete Galerkin scheme is uniquely solvable and provide asymptotic error estimates in terms of the space discretization parameter h. We also propose a fully discrete Galerkin scheme based on a backward Euler time stepping. Here again we provide error estimates that prove optimal convergence. Moreover, we obtain error estimates for the eddy currents and the magnetic induction field.

The chapter is organized as follows. In Section 3.2, we summarize some results from [31, 32, 34] concerning tangential traces in $\mathbf{H}(\mathbf{curl};\Omega)$ and recall some basic results for the study of time-dependent problems. In Section 3.3, we introduce the model problem and show how to handle the constraint satisfied by the electric field in the insulator by means of a Lagrange multiplier. In Section 3.4, we prove that the resulting saddle point problem is uniquely solvable. The derivation of a semi-discretization in space and its convergence analysis are reported in Section 3.5. Finally, a backward Euler method is employed to obtain a time discretization of the problem. The results presented in Section 3.6 prove that the resulting fully discrete scheme is convergent in an optimal way. We end this paper by summarizing its main results in Section 3.7.

3.2 Preliminaries

We use boldface letters to denote vectors as well as vector–valued functions and the symbol $|\cdot|$ represents the standard Euclidean norm for vectors. In this section Ω is a generic Lipschitz bounded domain of $\mathbb{R}^3$. We denote by Γ its boundary and by $\boldsymbol{n}$ the unit outward normal to Ω. Let

$$(f, g)_{0,\Omega} := \int_\Omega fg$$

be the inner product in $\mathrm{L}^2(\Omega)$ and $\|\cdot\|_{0,\Omega}$ the corresponding norm. As usual, for all $s > 0$, $\|\cdot\|_{s,\Omega}$ stands for the norm of the Hilbertian Sobolev space $\mathrm{H}^s(\Omega)$ and $|\cdot|_{s,\Omega}$ for the corresponding seminorm. The space $\mathrm{H}^{1/2}(\Gamma)$ is defined by localization on the Lipschitz surface Γ. We denote by $\|\cdot\|_{1/2,\Gamma}$ the norm in $\mathrm{H}^{1/2}(\Gamma)$ and $\langle\cdot,\cdot\rangle_\Gamma$ stands for the duality pairing between $\mathrm{H}^{1/2}(\Gamma)$ and its dual $\mathrm{H}^{-1/2}(\Gamma)$.

We denote by $\boldsymbol{\gamma} : \mathrm{H}^1(\Omega)^3 \to \mathrm{H}^{1/2}(\Gamma)^3$ the standard trace operator acting on vectors and define the tangential trace $\boldsymbol{\gamma}_\tau : \mathcal{C}^\infty(\overline{\Omega})^3 \to \mathrm{L}^2(\Gamma)^3$ as $\boldsymbol{q} \mapsto \boldsymbol{\gamma}\boldsymbol{q} \times \boldsymbol{n}$. Extending the tangential trace by completeness to $\mathrm{H}^1(\Omega)^3$, we define the space $\mathbf{H}_\perp^{1/2}(\Gamma) := \boldsymbol{\gamma}_\tau(\mathrm{H}^1(\Omega)^3)$

endowed with the norm

$$\|\boldsymbol{\lambda}\|_{\mathbf{H}_{\perp}^{1/2}(\Gamma)} := \inf_{\boldsymbol{q}\in \mathrm{H}^1(\Omega)^3} \{\|\boldsymbol{q}\|_{1,\Omega} : \ \boldsymbol{\gamma}_\tau(\boldsymbol{q}) = \boldsymbol{\lambda}\},$$

which makes the linear mapping $\boldsymbol{\gamma}_\tau : \mathrm{H}^1(\Omega)^3 \to \mathbf{H}_{\perp}^{1/2}(\Gamma)$ continuous and surjective. We refer to [31] for an intrinsic definition of $\mathbf{H}_{\perp}^{1/2}(\Gamma)$ in the case of a curvilinear Lipschitz polyhedron Ω. We introduce now the dual space $\mathbf{H}_{\perp}^{-1/2}(\Gamma)$ of $\mathbf{H}_{\perp}^{1/2}(\Gamma)$ with respect to the skew-symmetric pairing

$$\langle \boldsymbol{\lambda}, \boldsymbol{\eta}\rangle_{\tau,\Gamma} := \int_\Gamma (\boldsymbol{\lambda}\times \boldsymbol{n})\cdot \boldsymbol{\eta} \qquad \forall \boldsymbol{\lambda}, \boldsymbol{\eta} \in \mathrm{L}^2(\Gamma)^3.$$

The Green's formula

$$(\boldsymbol{u}, \mathbf{curl}\, \boldsymbol{v})_{0,\Omega} - (\mathbf{curl}\, \boldsymbol{u}, \boldsymbol{v})_{0,\Omega} = \langle \boldsymbol{\gamma}_\tau \boldsymbol{u}, \boldsymbol{\gamma}_\tau \boldsymbol{v}\rangle_{\tau,\Gamma} \quad \forall \boldsymbol{u} \in \mathcal{C}^\infty(\overline{\Omega})^3, \ \forall \boldsymbol{v} \in \mathrm{H}^1(\Omega)^3, \tag{3.1}$$

and the density of $\mathcal{C}^\infty(\overline{\Omega})^3$ in $\mathbf{H}(\mathbf{curl}, \Omega)$ (see [57, Theorem 3.26]) prove that

$$\boldsymbol{\gamma}_\tau : \ \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}_{\perp}^{-1/2}(\Gamma)$$

is continuous. A more accurate result is given by the following theorem.

Theorem 3.2.1 *Let*

$$\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) := \left\{\boldsymbol{\eta} \in \mathbf{H}_{\perp}^{-1/2}(\Gamma) : \ \mathrm{div}_\Gamma\, \boldsymbol{\eta} \in \mathrm{H}^{-1/2}(\Gamma)\right\}.$$

The operator $\boldsymbol{\gamma}_\tau : \ \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ *is continuous, surjective, and has a continuous right inverse.*

Proof. See [32, Theorem 4.6] for the case of Lipschitz polyhedra (the proper definition of div_Γ can be found in the same reference, as well). The more general case of Lipschitz domains is shown in [34, Theorem 4.1]. □

The kernel of the tangential trace operator $\boldsymbol{\gamma}_\tau$ in $\mathbf{H}(\mathbf{curl}, \Omega)$ is the closed subspace

$$\mathbf{H}_0(\mathbf{curl}, \Omega) := \{\boldsymbol{v} \in \mathbf{H}(\mathbf{curl}, \Omega) : \ \boldsymbol{\gamma}_\tau \boldsymbol{v} = 0\}.$$

We will also use the normal trace $\gamma_{\boldsymbol{n}} : \mathcal{C}^\infty(\overline{\Omega})^3 \to \mathrm{L}^2(\Gamma)$ given by $\boldsymbol{q} \mapsto \boldsymbol{\gamma q}\cdot \boldsymbol{n}$. It is well known that this operator can be extended to a continuous and surjective mapping (cf. [57, Theorem 3.24])

$$\gamma_{\boldsymbol{n}} : \ \mathbf{H}(\mathrm{div}, \Omega) \to \mathrm{H}^{-1/2}(\Gamma),$$

where $\mathbf{H}(\mathrm{div}, \Omega) := \{\boldsymbol{q} \in \mathrm{L}^2(\Omega)^3 : \ \mathrm{div}\, \boldsymbol{q} \in \mathrm{L}^2(\Omega)\}$ is endowed with the graph norm.

Since we will deal with a time-domain problem, besides the Sobolev spaces defined above, we need to introduce spaces of functions defined on a bounded time interval $(0, T)$ and with values in a separable Hilbert space V, whose norm is denoted here by $\|\cdot\|_V$. We use the notation $\mathcal{C}^0([0, T]; V)$ for the Banach space consisting of all continuous functions $f : [0, T] \to V$. More generally, for any $k \in \mathbb{N}$, $\mathcal{C}^k([0, T]; V)$ denotes the subspace of $\mathcal{C}^0([0, T]; V)$ of all functions f with (strong) derivatives in $\mathcal{C}^0([0, T]; V)$, *i.e.*

$$\mathcal{C}^k([0, T]; V) := \left\{ f \in \mathcal{C}^0([0, T]; V) : \frac{d^j f}{dt^j} \in \mathcal{C}^0([0, T]; V),\ 1 \leq j \leq k \right\}.$$

We also consider the space $\mathrm{L}^2(0, T; V)$ of classes of functions $f : (0, T) \to V$ that are Böchner-measurable and such that

$$\|f\|^2_{\mathrm{L}^2(0,T;V)} := \int_0^T \|f(t)\|_V^2 \, dt < +\infty.$$

Furthermore, we will use the space

$$\mathrm{H}^1(0, T; V) := \left\{ f \in \mathrm{L}^2(0, T; V) : \ \frac{d}{dt} f \in \mathrm{L}^2(0, T; V) \right\},$$

where $\frac{d}{dt} f$ is the (generalized) time derivative of f; see, for instance [73, Section 23.5]. In what follows, we will use indistinctly the notations

$$\frac{d}{dt} f = \partial_t f$$

to express the time derivative of f. Analogously, we define $\mathrm{H}^k(0, T; V)$ for all $k \in \mathbb{N}$.

3.3 Variational formulation

Our purpose is to determine the eddy currents induced in a three-dimensional conducting domain represented by the open and bounded set Ω_{c}, by a given time-dependent compactly-supported current density $\boldsymbol{J}$. We assume that Ω_{c} is a Lipschitz domain and denote by $\boldsymbol{n}$ the unit outward normal on $\Sigma := \partial\Omega_{\mathrm{c}}$. We denote by Σ_i, $i = 1, \cdots, I$, the connected components of Σ.

The electric and magnetic fields $\boldsymbol{E}(\boldsymbol{x}, t)$ and $\boldsymbol{H}(\boldsymbol{x}, t)$ are solutions of a submodel of

Maxwell's equations obtained by neglecting the displacement currents (see [13]):

$$\partial_t (\mu \boldsymbol{H}) + \mathbf{curl}\, \boldsymbol{E} = \mathbf{0} \qquad \text{in } \mathbb{R}^3 \times (0,T), \tag{3.2}$$

$$\mathbf{curl}\, \boldsymbol{H} = \boldsymbol{J} + \sigma \boldsymbol{E} \qquad \text{in } \mathbb{R}^3 \times [0,T), \tag{3.3}$$

$$\operatorname{div}(\varepsilon \boldsymbol{E}) = 0 \qquad \text{in } \left(\mathbb{R}^3 \setminus \Omega_{\mathrm{c}}\right) \times [0,T), \tag{3.4}$$

$$\int_{\Sigma_i} \varepsilon \boldsymbol{E} \cdot \boldsymbol{n} = 0 \qquad \text{in } [0,T), \quad i = 1, \cdots, I, \tag{3.5}$$

$$\boldsymbol{H}(\boldsymbol{x}, 0) = \boldsymbol{H}_0(\boldsymbol{x}) \qquad \text{in } \mathbb{R}^3, \tag{3.6}$$

$$\boldsymbol{H}(\boldsymbol{x}, t) = O\left(\frac{1}{|\boldsymbol{x}|}\right) \quad \text{and} \quad \boldsymbol{E}(\boldsymbol{x}, t) = O\left(\frac{1}{|\boldsymbol{x}|}\right) \qquad \text{as } |\boldsymbol{x}| \to \infty, \tag{3.7}$$

where the asymptotic behavior (3.7) holds uniformly in $(0,T)$. The electric permittivity ε, the electric conductivity σ, and the magnetic permeability μ are piecewise smooth real valued functions satisfying:

$$\varepsilon_1 \geq \varepsilon(\boldsymbol{x}) \geq \varepsilon_0 > 0 \quad \text{a.e. in } \Omega_{\mathrm{c}} \qquad \text{and} \qquad \varepsilon(\boldsymbol{x}) = \varepsilon_0 \quad \text{a.e. in } \mathbb{R}^3 \setminus \Omega_{\mathrm{c}}, \tag{3.8}$$

$$\sigma_1 \geq \sigma(\boldsymbol{x}) \geq \sigma_0 > 0 \quad \text{a.e. in } \Omega_{\mathrm{c}} \qquad \text{and} \qquad \sigma(\boldsymbol{x}) = 0 \quad \text{a.e. in } \mathbb{R}^3 \setminus \Omega_{\mathrm{c}}, \tag{3.9}$$

$$\mu_1 \geq \mu(\boldsymbol{x}, t) \geq \mu_0 > 0 \quad \text{a.e. in } \Omega_{\mathrm{c}} \times [0,T) \qquad \text{and} \qquad \mu(\boldsymbol{x}, t) = \mu_0 \quad \text{a.e. in } (\mathbb{R}^3 \setminus \Omega_{\mathrm{c}}) \times [0,T). \tag{3.10}$$

Notice that, as a consequence of (3.3) and (3.9), $\boldsymbol{J}$ is divergence-free in $\mathbb{R}^3 \setminus \Omega_{\mathrm{c}}$ and $\int_{\Sigma_i} \boldsymbol{J} \cdot \boldsymbol{n} = 0$, $i = 1, \cdots, I$, for all $t \in [0,T)$.

We will formulate our problem in terms of the time primitive of the electric field

$$\boldsymbol{u}(\boldsymbol{x}, t) := \int_0^t \boldsymbol{E}(\boldsymbol{x}, s)\, ds.$$

To this end, we integrate (3.2) with respect to t,

$$\mu(\boldsymbol{x}, t) \boldsymbol{H}(\boldsymbol{x}, t) = -\,\mathbf{curl}\, \boldsymbol{u}(\boldsymbol{x}, t) + \mu(\boldsymbol{x}, 0) \boldsymbol{H}_0, \tag{3.11}$$

and use the resulting expression of the magnetic field in (3.3) to obtain

$$\sigma \partial_t \boldsymbol{u} + \mathbf{curl}\left(\frac{1}{\mu}\, \mathbf{curl}\, \boldsymbol{u}\right) = \boldsymbol{f},$$

where

$$\boldsymbol{f}(\boldsymbol{x}, t) := \mathbf{curl}\left(\frac{\mu(\boldsymbol{x}, 0)}{\mu(\boldsymbol{x}, t)} \boldsymbol{H}_0\right) - \boldsymbol{J}(\boldsymbol{x}, t).$$

Let us remark that the support of $\boldsymbol{f}$ is compact, since (3.3) is assumed to hold at $t = 0$.

Notice that as a consequence of the decay conditions (3.7), we may assume that the electromagnetic field is weak far away from $\Omega_{\rm c}$. Motivated by this fact, and aiming to obtaining a suitable simplification of our model problem, we introduce a closed surface Γ located sufficiently far from $\overline{\Omega}_{\rm c}$ and assume that $\boldsymbol{u}$ has a vanishing tangential trace on this surface. Hence, we will formulate our problem in the bounded domain Ω with boundary Γ. We assume that Ω is simply connected, with a connected boundary, and that it contains $\Omega_{\rm c}$ and the support of $\boldsymbol{J}$ (and, consequently, the support of $\boldsymbol{f}$). We define $\Omega_{\rm d} := \Omega \setminus \overline{\Omega}_{\rm c}$.

The last considerations lead us to the following formulation of the eddy current problem:
Find $\boldsymbol{u} : \Omega \times [0, T) \to \mathbb{R}^3$ such that:

$$\begin{aligned} \sigma \partial_t \boldsymbol{u} + \mathbf{curl}\left(\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{u}\right) &= \boldsymbol{f} && \text{in } \Omega \times (0,T), \\ \operatorname{div}(\varepsilon \boldsymbol{u}) &= 0 && \text{in } \Omega_{\rm d} \times [0,T), \\ \langle \gamma_n(\varepsilon \boldsymbol{u}), 1\rangle_{\Sigma_i} &= 0 && \text{in } [0,T), \quad i = 1,\dots,I, \\ \boldsymbol{\gamma}_\tau \boldsymbol{u} &= \mathbf{0} && \text{on } \Gamma \times [0,T), \\ \boldsymbol{u}(\cdot, 0) &= \mathbf{0} && \text{in } \Omega. \end{aligned} \tag{3.12}$$

It is important to notice here that, due to (3.3) and (3.9), the data $\boldsymbol{f}$ satisfies the compatibility conditions

$$\operatorname{div} \boldsymbol{f} = 0 \text{ in } \Omega_{\rm d} \qquad \text{and} \qquad \langle \gamma_n \boldsymbol{f}, 1\rangle_{\Sigma_i} = 0, \quad i = 1, 2, \dots, I, \tag{3.13}$$

for all $t \in (0, T)$.

We introduce the space

$$M(\Omega_{\rm d}) := \left\{ \vartheta \in \mathrm{H}^1(\Omega_{\rm d}) : \ \gamma\vartheta|_\Gamma = 0 \text{ and } \gamma\vartheta|_{\Sigma_i} = C_i, \ i = 1, \dots, I \right\},$$

where C_i, $i = 1, \dots, I$, are arbitrary constants. The Poincaré inequality shows that $|\cdot|_{1,\Omega_{\rm d}}$ is a norm on $M(\Omega_{\rm d})$ equivalent to the usual $\mathrm{H}^1(\Omega_{\rm d})$-norm. Next, let

$$V_0(\Omega) := \{ \boldsymbol{v} \in \mathbf{H}_0(\mathbf{curl}, \Omega) : \ b(\boldsymbol{v}, \vartheta) = 0 \quad \forall \vartheta \in M(\Omega_{\rm d}) \}, \tag{3.14}$$

where

$$b(\boldsymbol{u}, \vartheta) := (\varepsilon \boldsymbol{u}, \mathbf{grad}\,\vartheta)_{0,\Omega_{\rm d}}.$$

Lemma 3.3.1 *There holds*

$$V_0(\Omega) = \{\boldsymbol{v} \in \mathbf{H}_0(\mathbf{curl}, \Omega) : \ \operatorname{div}(\varepsilon \boldsymbol{v}) = 0 \ \textit{in} \ \Omega_{\mathrm{d}}, \ \langle \gamma_n(\varepsilon \boldsymbol{v}), 1 \rangle_{\Sigma_i} = 0, \ i = 1, \cdots, I\}.$$

Proof. If $\boldsymbol{v} \in V_0(\Omega)$, then, in particular,

$$b(\boldsymbol{v}, \vartheta) = 0 \qquad \forall \vartheta \in \mathcal{D}(\Omega_{\mathrm{d}}),$$

where $\mathcal{D}(\Omega_{\mathrm{d}})$ is the space of infinitely differentiable functions with compact support in Ω_{d}. This implies that $\operatorname{div}(\varepsilon \boldsymbol{v}) = 0$ in Ω_{d}. Choosing now $\vartheta_i \in M(\Omega_{\mathrm{d}})$ such that $\gamma \vartheta_i|_{\Sigma_j} = \delta_{i,j}$ for $1 \le i, j \le I$, we obtain from a Green's formula that

$$0 = b(\boldsymbol{v}, \vartheta_i) = \langle \gamma_n(\varepsilon \boldsymbol{v}), 1 \rangle_{\Sigma_i}.$$

The other inclusion is straightforward. $\square$

By testing the first equation of (3.12) with a function $\boldsymbol{v} \in V_0(\Omega)$ and using (3.1), we obtain the following variational formulation:
Find $\boldsymbol{u} \in \mathcal{W}_0$ such that

$$\begin{aligned} \frac{d}{dt}(\sigma \boldsymbol{u}(t), \boldsymbol{v})_{0,\Omega_{\mathrm{c}}} + \left(\frac{1}{\mu(t)} \mathbf{curl}\, \boldsymbol{u}(t), \mathbf{curl}\, \boldsymbol{v} \right)_{0,\Omega} &= (\boldsymbol{f}(t), \boldsymbol{v})_{0,\Omega} \qquad \forall \boldsymbol{v} \in V_0(\Omega), \\ \boldsymbol{u}|_{\Omega_{\mathrm{c}}}(0) &= \mathbf{0}, \end{aligned} \tag{3.15}$$

where

$$\mathcal{W}_0 := \{\boldsymbol{v} \in \mathrm{L}^2(0,T; V_0(\Omega)) : \ \boldsymbol{v}|_{\Omega_{\mathrm{c}}} \in W^1(0,T; \mathbf{H}(\mathbf{curl}, \Omega_{\mathrm{c}}))\},$$

with

$$\begin{aligned} &W^1(0,T; \mathbf{H}(\mathbf{curl}, \Omega_{\mathrm{c}})) \\ &\qquad := \left\{\boldsymbol{v} \in \mathrm{L}^2(0,T; \mathbf{H}(\mathbf{curl}, \Omega_{\mathrm{c}})) : \ \partial_t \boldsymbol{v} \in \mathrm{L}^2(0,T; \mathbf{H}(\mathbf{curl}, \Omega_{\mathrm{c}})')\right\}. \end{aligned}$$

Here, $\mathbf{H}(\mathbf{curl}, \Omega_{\mathrm{c}})'$ is the dual space of $\mathbf{H}(\mathbf{curl}, \Omega_{\mathrm{c}})$ with respect to the pivot space

$$\mathrm{L}^2(\Omega_{\mathrm{c}}, \sigma)^3 := \left\{ \boldsymbol{v} : \Omega_{\mathrm{c}} \to \mathbb{R}^3 \text{ Lebesgue-measurable} : \int_{\Omega_{\mathrm{c}}} \sigma |\boldsymbol{v}|^2 < \infty \right\}.$$

Notice that the initial condition makes sense thanks to the continuous embedding (see for instance [73, Proposition 23.23])

$$W^1(0,T; \mathbf{H}(\mathbf{curl}, \Omega_{\mathrm{c}})) \hookrightarrow \mathcal{C}^0(0,T; \mathrm{L}^2(\Omega_{\mathrm{c}}, \sigma)^3).$$

In order to avoid the task of constructing a conforming finite element discretization of (3.15), we take advantage of Lemma 3.3.1 and propose a mixed formulation of the problem. To this end, we relax as follows the divergence-free restriction through a Lagrange multiplier:
Find $\boldsymbol{u} \in \mathcal{W}$ and $\lambda \in \mathrm{L}^2(0,T;M(\Omega_{\mathrm{d}}))$ such that

$$\begin{aligned} \frac{d}{dt}\left[(\boldsymbol{u}(t),\boldsymbol{v})_\sigma + b(\boldsymbol{v},\lambda(t))\right] + a(t;\boldsymbol{u}(t),\boldsymbol{v}) = (\boldsymbol{f}(t),\boldsymbol{v})_{0,\Omega} \qquad \forall \boldsymbol{v} \in \mathbf{H}_0(\mathbf{curl},\Omega), \\ b(\boldsymbol{u}(t),\vartheta) = 0 \qquad \forall \vartheta \in M(\Omega_{\mathrm{d}}), \\ \boldsymbol{u}|_{\Omega_{\mathrm{c}}}(0) = \mathbf{0}, \end{aligned} \tag{3.16}$$

where

$$\mathcal{W} := \left\{ \boldsymbol{v} \in \mathrm{L}^2(0,T;\mathbf{H}_0(\mathbf{curl},\Omega)) : \ \boldsymbol{v}|_{\Omega_{\mathrm{c}}} \in W^1(0,T;\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}})) \right\},$$

$$(\boldsymbol{u},\boldsymbol{v})_\sigma := (\sigma\boldsymbol{u},\boldsymbol{v})_{0,\Omega_{\mathrm{c}}}, \qquad \text{and} \qquad a(t;\boldsymbol{u},\boldsymbol{v}) := \left(\frac{1}{\mu(t)}\,\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v}\right)_{0,\Omega}.$$

Notice that $\mathcal{W}$, endowed with the graph norm

$$\|\boldsymbol{v}\|_{\mathcal{W}}^2 := \int_0^T \|\boldsymbol{v}(t)\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2\,dt + \int_0^T \|\partial_t \boldsymbol{v}(t)\|_{\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}})'}^2\,dt,$$

is a Hilbert space and that $\mathcal{W}_0$ is a closed subspace of $\mathcal{W}$.

3.4 Existence and uniqueness

We introduce the space

$$V_0(\Omega_{\mathrm{d}}) := \{\boldsymbol{v} \in \mathbf{H}_0(\mathbf{curl},\Omega_{\mathrm{d}}) : \ b(\boldsymbol{v},\vartheta) = 0 \quad \forall \vartheta \in M(\Omega_{\mathrm{d}})\} \tag{3.17}$$

and recall the following result.

Lemma 3.4.1 *The seminorm $\boldsymbol{v} \mapsto \|\mathbf{curl}\,\boldsymbol{v}\|_{0,\Omega_{\mathrm{d}}}$ is a norm on $V_0(\Omega_{\mathrm{d}})$ equivalent to the usual norm of $\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{d}})$; i.e. there exists a constant $C > 0$ depending only on Ω such that*

$$\|\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{d}})} \leq C\|\mathbf{curl}\,\boldsymbol{v}\|_{0,\Omega_{\mathrm{d}}} \qquad \forall \boldsymbol{v} \in V_0(\Omega_{\mathrm{d}}).$$

Proof. See, for instance, [45, Corollary 4.4]. □

Lemma 3.4.2 *The linear mapping*

$$\begin{aligned} \mathcal{E}: \ \mathbf{H}(\mathbf{curl},\Omega_{\rm c}) &\rightarrow V_0(\Omega) \\ \boldsymbol{v}_{\rm c} &\mapsto \mathcal{E}\boldsymbol{v}_{\rm c} \end{aligned}$$

characterized by $(\mathcal{E}\boldsymbol{v}_{\rm c})\,|_{\Omega_{\rm c}} = \boldsymbol{v}_{\rm c}$ *and*

$$(\mathbf{curl}\,\mathcal{E}\boldsymbol{v}_{\rm c}, \mathbf{curl}\,\boldsymbol{w})_{0,\Omega_{\rm d}} = 0 \qquad \forall \boldsymbol{w} \in V_0(\Omega_{\rm d}) \tag{3.18}$$

is well defined and bounded.

Proof. Let us denote here by $\boldsymbol{\gamma}_\tau^{\rm c}$ and $\boldsymbol{\gamma}_\tau^{\rm d}$ the tangential traces on Σ taken from $\Omega_{\rm c}$ and $\Omega_{\rm d}$, respectively. We know from Theorem 3.2.1 that there exists a continuous right inverse of the tangential trace operator $\boldsymbol{\gamma}_\tau^{\rm d}$:

$$(\boldsymbol{\gamma}_\tau^{\rm d})^{-1}: \ \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Sigma) \rightarrow \{\boldsymbol{v}|_{\Omega_{\rm d}}: \ \boldsymbol{v} \in \mathbf{H}_0(\mathbf{curl},\Omega)\}\,.$$

It follows that the linear operator

$$\begin{aligned} \mathcal{L}: \ \mathbf{H}(\mathbf{curl},\Omega_{\rm c}) &\rightarrow \{\boldsymbol{v}|_{\Omega_{\rm d}}: \ \boldsymbol{v} \in \mathbf{H}_0(\mathbf{curl},\Omega)\} \\ \boldsymbol{v}_{\rm c} &\mapsto \mathcal{L}\boldsymbol{v}_{\rm c} := (\boldsymbol{\gamma}_\tau^{\rm d})^{-1}(\boldsymbol{\gamma}_\tau^{\rm c}\boldsymbol{v}_{\rm c}) \end{aligned} \tag{3.19}$$

is bounded, namely,

$$\|\mathcal{L}\boldsymbol{v}_{\rm c}\|_{\mathbf{H}(\mathbf{curl},\Omega_{\rm d})} \leq C_0 \|\boldsymbol{v}_{\rm c}\|_{\mathbf{H}(\mathbf{curl},\Omega_{\rm c})} \qquad \forall \boldsymbol{v}_{\rm c} \in \mathbf{H}(\mathbf{curl},\Omega_{\rm c}), \tag{3.20}$$

and it satisfies $\boldsymbol{\gamma}_\tau^{\rm d}(\mathcal{L}\boldsymbol{v}_{\rm c}) = \boldsymbol{\gamma}_\tau^{\rm c}\boldsymbol{v}_{\rm c}$ on Σ. Notice that $\mathcal{L}\boldsymbol{v}_{\rm c}$ is an $\mathbf{H}(\mathbf{curl};\Omega)$-conforming extension of $\boldsymbol{v}_{\rm c}$ to the whole Ω, but it does not necessarily fulfill (3.18).

Given $\boldsymbol{v}_{\rm c} \in \mathbf{H}(\mathbf{curl},\Omega_{\rm c})$, consider the problem of finding $\boldsymbol{z} \in \mathcal{L}\boldsymbol{v}_{\rm c} + \mathbf{H}_0(\mathbf{curl},\Omega_{\rm d})$ and $\rho \in M(\Omega_{\rm d})$ satisfying

$$\begin{aligned} (\mathbf{curl}\,\boldsymbol{z}, \mathbf{curl}\,\boldsymbol{w})_{0,\Omega_{\rm d}} + b(\boldsymbol{w},\rho) &= 0 \qquad \forall \boldsymbol{w} \in \mathbf{H}_0(\mathbf{curl},\Omega_{\rm d}), \\ b(\boldsymbol{z},\vartheta) &= 0 \qquad \forall \vartheta \in M(\Omega_{\rm d}). \end{aligned}$$

The well-posedness of this problem is guaranteed by the Babuška-Brezzi theory. Indeed, on the one hand, the fact that $\mathbf{grad}(M(\Omega_{\rm d})) \subset \mathbf{H}_0(\mathbf{curl},\Omega_{\rm d})$ implies easily the following inf-sup condition for b:

$$\sup_{\boldsymbol{z}\in\mathbf{H}_0(\mathbf{curl},\Omega_{\rm d})} \frac{b(\boldsymbol{z},\vartheta)}{\|\boldsymbol{z}\|_{\mathbf{H}(\mathbf{curl},\Omega_{\rm d})}} \geq \varepsilon_0 \frac{(\mathbf{grad}\,\vartheta, \mathbf{grad}\,\vartheta)_{0,\Omega_{\rm d}}}{\|\,\mathbf{grad}\,\vartheta\|_{\mathbf{H}(\mathbf{curl},\Omega_{\rm d})}} = \varepsilon_0 |\vartheta|_{1,\Omega_{\rm d}} \qquad \forall \vartheta \in M(\Omega_{\rm d}).$$

On the other hand, Lemma 3.4.1 ensures the ellipticity in the kernel property: there exists $C_1 > 0$ such that

$$(\mathbf{curl}\,\boldsymbol{w}, \mathbf{curl}\,\boldsymbol{w})_{0,\Omega_\mathrm{d}} \geq C_1 \|\boldsymbol{w}\|^2_{\mathbf{H}_0(\mathbf{curl},\Omega_\mathrm{d})} \qquad \forall \boldsymbol{w} \in V_0(\Omega_\mathrm{d}). \tag{3.21}$$

It is now clear that $\mathcal{E}\boldsymbol{v}_\mathrm{c} := \boldsymbol{z}$ satisfies (3.18) and $(\mathcal{E}\boldsymbol{v}_\mathrm{c})\,|_{\Omega_\mathrm{c}} = \boldsymbol{v}_\mathrm{c}$. The uniqueness of solution of (3.18) follows from (3.21). Moreover, by virtue of the stability results provided by the Babuška-Brezzi theory, there exists a constant $C_2 > 0$ such that

$$\|\mathcal{E}\boldsymbol{v}_\mathrm{c}\|_{\mathbf{H}(\mathbf{curl},\Omega_\mathrm{d})} \leq C_2 \|\mathcal{L}\boldsymbol{v}_\mathrm{c}\|_{\mathbf{H}(\mathbf{curl},\Omega_\mathrm{d})}.$$

Finally, (3.20) yields the estimate

$$\|\mathcal{E}\boldsymbol{v}_\mathrm{c}\|_{\mathbf{H}(\mathbf{curl};\Omega)} \leq \sqrt{1 + (C_0 C_2)^2}\|\boldsymbol{v}_\mathrm{c}\|_{\mathbf{H}(\mathbf{curl},\Omega_\mathrm{c})} \qquad \forall \boldsymbol{v}_\mathrm{c} \in \mathbf{H}(\mathbf{curl}, \Omega_\mathrm{c}).$$

□

Lemma 3.4.3 *The inner product in $V_0(\Omega)$*

$$(\boldsymbol{u}, \boldsymbol{v})_{V_0(\Omega)} := (\boldsymbol{u}, \boldsymbol{v})_\sigma + (\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})_{0,\Omega} \tag{3.22}$$

induces a norm $\|\cdot\|_{V_0(\Omega)}$ that is equivalent to the $\mathbf{H}(\mathbf{curl};\Omega)$ norm. Moreover, the following decomposition is orthogonal with respect to the inner product $(\cdot,\cdot)_{V_0(\Omega)}$:

$$V_0(\Omega) = \widetilde{V_0(\Omega_\mathrm{d})} \oplus \mathcal{E}(\mathbf{H}(\mathbf{curl}, \Omega_\mathrm{c})), \tag{3.23}$$

where $\widetilde{V_0(\Omega_\mathrm{d})}$ is the subspace of $V_0(\Omega)$ obtained by extending by zero the functions of $V_0(\Omega_\mathrm{d})$ to the whole domain Ω.

Proof. For any $\boldsymbol{v} \in V_0(\Omega)$, let us denote $\boldsymbol{v}_\mathrm{c} := \boldsymbol{v}|_{\Omega_\mathrm{c}}$. Notice that $\boldsymbol{v} - \mathcal{E}\boldsymbol{v}_\mathrm{c} \in \widetilde{V_0(\Omega_\mathrm{d})}$. The triangle inequality and Lemma 3.4.1 ensure the existence of a constant $C_0 > 0$ such that

$$\|\boldsymbol{v}\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} \leq 2C_0^2 \|\mathbf{curl}(\boldsymbol{v} - \mathcal{E}\boldsymbol{v}_\mathrm{c})\|^2_{0,\Omega_\mathrm{d}} + 2\|\mathcal{E}\boldsymbol{v}_\mathrm{c}\|^2_{\mathbf{H}(\mathbf{curl};\Omega)}.$$

Hence, using again the triangle inequality and Lemma 3.4.2, we have

$$\|\boldsymbol{v}\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} \leq C_1 \left(\|\mathbf{curl}\,\boldsymbol{v}\|^2_{0,\Omega_\mathrm{d}} + \|\boldsymbol{v}_\mathrm{c}\|^2_{\mathbf{H}(\mathbf{curl},\Omega_\mathrm{c})}\right) = C_1 \left(\|\boldsymbol{v}\|^2_{0,\Omega_\mathrm{c}} + \|\mathbf{curl}\,\boldsymbol{v}\|^2_{0,\Omega}\right).$$

Consequently,

$$\|\boldsymbol{v}\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} \leq C_1 \max\{\sigma_0^{-1}, 1\}\, \|\boldsymbol{v}\|^2_{V_0(\Omega)}.$$

The other inequality is straightforward.

Finally, it is easy to check that $\mathcal{E}(\mathbf{H}(\mathbf{curl},\Omega_c))$ is the orthogonal complement of $\widetilde{V_0(\Omega_d)}$ in $V_0(\Omega)$ with respect to the inner product $(\cdot,\cdot)_{V_0(\Omega)}$. $\square$

We are now in a position to prove the main result of this section.

Theorem 3.4.1 *Problem* (3.16) *has a unique solution* $(\boldsymbol{u},\lambda)$. *Furthermore, there exists* $C>0$ *such that*

$$\max_{t\in[0,T]}\|\boldsymbol{u}(t)\|^2_{0,\Omega_c}+\int_0^T\|\boldsymbol{u}(t)\|^2_{\mathbf{H}(\mathbf{curl};\Omega)}\,dt\le C\int_0^T\|\boldsymbol{f}(t)\|^2_{0,\Omega}\,dt. \tag{3.24}$$

Proof. We first notice that the second equation of (3.16) means that $\boldsymbol{u}\in\mathcal{W}_0$. The decomposition (3.23) implies that the direct sum

$$\mathcal{W}_0=\mathrm{L}^2(0,T;\widetilde{V_0(\Omega_d)})\oplus\mathcal{E}(W^1(0,T;\mathbf{H}(\mathbf{curl},\Omega_c)))$$

is orthogonal with respect to the inner product $\int_0^T(\cdot,\cdot)_{V_0(\Omega)}\,dt$. Hence $\boldsymbol{u}=\boldsymbol{u}_d+\mathcal{E}\boldsymbol{u}_c$, with $\boldsymbol{u}_d\in\mathrm{L}^2(0,T;\widetilde{V_0(\Omega_d)})$ and $\mathcal{E}\boldsymbol{u}_c\in\mathcal{E}(W^1(0,T;\mathbf{H}(\mathbf{curl},\Omega_c)))$. Testing the first equation of (3.16) with $\boldsymbol{v}\in\widetilde{V_0(\Omega_d)}$, we find that the first component satisfies

$$\left(\frac{1}{\mu(t)}\,\mathbf{curl}\,\boldsymbol{u}_d(t),\mathbf{curl}\,\boldsymbol{v}\right)_{0,\Omega_d}=(\boldsymbol{f}(t),\boldsymbol{v})_{0,\Omega_d}\qquad\forall\boldsymbol{v}\in V_0(\Omega_d).$$

Lemma 3.4.1 and the Lax-Milgram lemma prove that this problem admits a unique solution and there exists $C_1>0$ such that

$$\int_0^T\|\boldsymbol{u}_d\|^2_{\mathbf{H}(\mathbf{curl},\Omega_d)}\,dt\le C_1\int_0^T\|\boldsymbol{f}(t)\|^2_{0,\Omega}\,dt. \tag{3.25}$$

The other component is determined by solving

$$\begin{gathered}\frac{d}{dt}(\boldsymbol{u}_c(t),\boldsymbol{v})_\sigma+a(t;\mathcal{E}\boldsymbol{u}_c(t),\mathcal{E}\boldsymbol{v})=(\boldsymbol{f}(t),\mathcal{E}\boldsymbol{v})_{0,\Omega}\qquad\forall\boldsymbol{v}\in\mathbf{H}(\mathbf{curl},\Omega_c),\\ \boldsymbol{u}_c(0)=\mathbf{0}.\end{gathered} \tag{3.26}$$

For any $t\in(0,T)$, the bilinear form $a(t;\mathcal{E}\cdot,\mathcal{E}\cdot)$ is clearly continuous and coercive on $\mathbf{H}(\mathbf{curl},\Omega_c)$:

$$a(t;\mathcal{E}\boldsymbol{v},\mathcal{E}\boldsymbol{v})+(\boldsymbol{v},\boldsymbol{v})_\sigma\ge\min\{\sigma_0,\mu_1^{-1}\}\|\boldsymbol{v}\|^2_{\mathbf{H}(\mathbf{curl},\Omega_c)}\qquad\forall\boldsymbol{v}\in\mathbf{H}(\mathbf{curl},\Omega_c).$$

Therefore, the well-posedness of the parabolic problem (3.26) follows immediately from a simple variant of Lions Theorem (see, for instance, [73, Corollary 23.26]). In addition, there exists $C_2>0$ such that

$$\max_{t\in[0,T]}\|\boldsymbol{u}_c(t)\|^2_{0,\Omega_c}+\int_0^T\|\boldsymbol{u}_c(t)\|^2_{\mathbf{H}(\mathbf{curl},\Omega_d)}\,dt\le C_2\int_0^T\|\boldsymbol{f}(t)\|^2_{0,\Omega}\,dt,$$

which, combined with (3.25) and the boundedness of $\mathcal{E}$, yields (3.24).

It remains to prove the existence and uniqueness of the Lagrange multiplier λ. Given $\vartheta \in M(\Omega_{\mathrm{d}})$, we denote by $\widetilde{\mathbf{grad}}\,\vartheta \in \mathbf{H}_0(\mathbf{curl},\Omega)$ the extension by zero of $\mathbf{grad}\,\vartheta$ to the whole Ω. Notice that the bilinear form b satisfies the inf-sup condition

$$\sup_{\boldsymbol{v}\in\mathbf{H}_0(\mathbf{curl},\Omega)} \frac{b(\boldsymbol{v},\vartheta)}{\|\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)}} \geq \frac{b(\widetilde{\mathbf{grad}}\,\vartheta,\vartheta)}{\|\widetilde{\mathbf{grad}}\,\vartheta\|_{\mathbf{H}(\mathbf{curl};\Omega)}} = \varepsilon_0|\vartheta|_{1,\Omega_{\mathrm{d}}} \qquad \forall\vartheta\in M(\Omega_{\mathrm{d}}). \tag{3.27}$$

Let us consider now $\mathcal{G}\in\mathcal{C}^0([0,T],\mathbf{H}_0(\mathbf{curl},\Omega)')$ defined by

$$\langle\mathcal{G}(t),\boldsymbol{v}\rangle := -\left(\boldsymbol{u}(t),\boldsymbol{v}\right)_\sigma - \int_0^t a(s;\boldsymbol{u}(s),\boldsymbol{v})\,ds + \int_0^t \left(\boldsymbol{f}(s),\boldsymbol{v}\right)_{0,\Omega}\,ds$$

for all $\boldsymbol{v}\in\mathbf{H}_0(\mathbf{curl},\Omega)$. By integrating the first equation of (3.15) with respect to t and using the second one, we obtain

$$\langle\mathcal{G}(t),\boldsymbol{v}\rangle = 0 \qquad \forall\boldsymbol{v}\in V_0(\Omega).$$

Therefore, taking into account the definition (3.14) of $V_0(\Omega)$, the inf-sup condition (3.27) guarantees the existence of a unique $\lambda(t)\in M(\Omega_{\mathrm{d}})$ such that (see [43, Lemma I.4.1])

$$b(\boldsymbol{v},\lambda(t)) = \langle\mathcal{G}(t),\boldsymbol{v}\rangle \qquad \forall\boldsymbol{v}\in\mathbf{H}_0(\mathbf{curl},\Omega). \tag{3.28}$$

We conclude that $(\boldsymbol{u},\lambda)$ solves (3.16) by differentiating the last identity with respect to t in the sense of distributions. □

The reason for which we have skipped the stability estimate for the Lagrange multiplier λ in the last theorem becomes clear from the following result.

Lemma 3.4.4 *The Lagrange multiplier* λ *of problem* (3.16) *vanishes identically.*

Proof. By virtue of the compatibility conditions (3.13),

$$(\boldsymbol{f},\mathbf{grad}\,\vartheta)_{0,\Omega_{\mathrm{d}}} = \langle\gamma_n\boldsymbol{f},\vartheta\rangle_{\partial\Omega_{\mathrm{d}}} = \sum_{i=1}^{I}\vartheta|_{\Sigma_i}\langle\gamma_n\boldsymbol{f},1\rangle_{\Sigma_i} = 0 \qquad \forall\vartheta\in M(\Omega_{\mathrm{d}}). \tag{3.29}$$

Consequently, testing the first equation of (3.16) with $\mathbf{grad}\,\vartheta$ (extended by zero to the whole Ω) yields

$$\frac{d}{dt}b(\mathbf{grad}\,\vartheta,\lambda(t)) = \left(\boldsymbol{f}(t),\mathbf{grad}\,\vartheta\right)_{0,\Omega_{\mathrm{d}}} = 0 \qquad \forall\vartheta\in M(\Omega_{\mathrm{d}}).$$

Next, we take $t=0$ in (3.28) and use the fact that $\mathcal{G}(0)=\mathbf{0}$ to deduce that $t\mapsto b(\mathbf{grad}\,\vartheta,\lambda(t))$ vanishes identically in $[0,T]$ for all $\vartheta\in M(\Omega_{\mathrm{d}})$. In particular $\varepsilon_0|\lambda(t)|^2_{1,\Omega_{\mathrm{d}}} = b(\mathbf{grad}\,\lambda(t),\lambda(t)) = 0$ for all $t\in[0,T]$, and the result follows. □

3.5 Analysis of the semi-discrete scheme.

In what follows we assume that Ω and Ω_c are Lipschitz polyhedra. Let $\{\mathcal{T}_h\}_h$ be a regular family of tetrahedral meshes of Ω such that each element $K \in \mathcal{T}_h$ is contained either in $\overline{\Omega}_c$ or in $\overline{\Omega}_d$. As usual, h stands for the largest diameter of the tetrahedra K in $\mathcal{T}_h$. Furthermore, we suppose that the family of triangulations $\{\mathcal{T}_h(\Sigma)\}_h$ induced by $\{\mathcal{T}_h\}_h$ on Σ is quasi-uniform.

We define a semi-discrete version of (3.16) by means of Nédélec finite elements. The local representation of the mth-order element of this family on a tetrahedron K is given by (see [57, Section 5.5])

$$\mathcal{N}_m(K) := \mathbb{P}_{m-1}^3 \oplus S_m,$$

where $\mathbb{P}_m$ is the set of polynomials of degree not greater than m and

$$S_m := \left\{ p \in \widetilde{\mathbb{P}}_m^3 : \ \boldsymbol{x} \cdot p(\boldsymbol{x}) = 0 \right\},$$

with $\widetilde{\mathbb{P}}_m$ being the set of homogeneous polynomials of degree m. The degrees of freedom of $\mathcal{N}_m(K)$ are given by

$$\mathcal{M}_1(\boldsymbol{v}) := \left\{ \int_E \boldsymbol{v} \cdot \boldsymbol{t}_e \, q \text{ for all } q \in \mathbb{P}_{m-1} \text{ for the six edges } E \text{ of } K \right\}, \tag{3.30}$$

where $\boldsymbol{t}_E$ is a unit tangent vector along E; when $m \geq 2$ one has to add

$$\mathcal{M}_2(\boldsymbol{v}) := \left\{ \int_F (\boldsymbol{v} \times \boldsymbol{n}) \cdot \boldsymbol{q} \text{ for all } \boldsymbol{q} \in \mathbb{P}_{m-2}^2 \text{ for the four faces } F \text{ of } K \right\}; \tag{3.31}$$

and finally for $m \geq 3$ one has to take also

$$\mathcal{M}_3(\boldsymbol{v}) := \left\{ \int_K \boldsymbol{v} \cdot \boldsymbol{q} \text{ for all } \boldsymbol{q} \in \mathbb{P}_{m-3}^3 \right\}. \tag{3.32}$$

Nédélec [61] has proved that these degrees of freedom are "curl-conforming" and determine a unique element of $\mathcal{N}_m(K)$. Then, for any smooth enough function $\boldsymbol{v}$ on K such that the moments (3.30)-(3.32) are well defined, we can define $\mathcal{I}_K \boldsymbol{v} \in X_h(\Omega)$ characterized by

$$\mathcal{M}_i(\boldsymbol{v}) = \mathcal{M}_i(\mathcal{I}_K \boldsymbol{v}) \qquad i = 1, 2, 3.$$

The corresponding global space $X_h(\Omega)$ is the space of functions that are locally in $\mathcal{N}_m(K)$ and have continuous tangential components across the faces of the triangulation $\mathcal{T}_h$:

$$X_h(\Omega) := \{ \boldsymbol{v} \in \mathbf{H}_0(\mathbf{curl}, \Omega) : \ \boldsymbol{v}|_K \in \mathcal{N}_m(K) \ \forall K \in \mathcal{T} \}.$$

We use standard mth-order Lagrange finite elements to approximate $M(\Omega_{\rm d})$:

$$M_h(\Omega_{\rm d}) := \left\{\vartheta \in {\rm H}^1(\Omega_{\rm d}) : \ \vartheta|_K \in \mathbb{P}_m \ \forall K \in \mathcal{T}, \ \vartheta|_\Gamma = 0, \ \vartheta|_{\Sigma_i} = C_i, \ i = 1, \dots, I\right\}.$$

We introduce the following semi-discretization of problem (3.16):
Find $\boldsymbol{u}_h(t) : [0,T] \to X_h(\Omega)$ and $\lambda_h(t) : [0,T] \to M_h(\Omega_{\rm d})$ such that

$$\begin{aligned} \frac{d}{dt}\left[(\boldsymbol{u}_h(t), \boldsymbol{v})_\sigma + b(\boldsymbol{v}, \lambda_h(t))\right] + a(t; \boldsymbol{u}_h(t), \boldsymbol{v}) = (\boldsymbol{f}(t), \boldsymbol{v})_{0,\Omega} \qquad & \forall \boldsymbol{v} \in X_h(\Omega), \\ b(\boldsymbol{u}_h(t), \vartheta) = 0 \qquad & \forall \vartheta \in M_h(\Omega_{\rm d}), \\ \boldsymbol{u}_h|_{\Omega_{\rm c}}(0) = \mathbf{0}. & \end{aligned} \tag{3.33}$$

Notice that the discrete kernel

$$V_{0,h}(\Omega) := \{\boldsymbol{v} \in X_h(\Omega) : \ b(\boldsymbol{v}, \vartheta) = 0 \ \forall \vartheta \in M_h(\Omega_{\rm d})\}$$

is not necessarily a subspace of $V_0(\Omega)$. We introduce

$$V_{0,h}(\Omega_{\rm d}) := \{\boldsymbol{v}|_{\Omega_{\rm d}} : \ \boldsymbol{v} \in V_{0,h}(\Omega)\} \cap \mathbf{H}_0(\mathbf{curl}, \Omega_{\rm d})$$

and recall the discrete analogue of Lemma 3.4.1.

Lemma 3.5.1 *The mapping $\boldsymbol{v} \mapsto \|\mathbf{curl}\, \boldsymbol{v}\|_{0,\Omega_{\rm d}}$ is a norm on $V_{0,h}(\Omega_{\rm d})$ uniformly equivalent to the $\mathbf{H}(\mathbf{curl}, \Omega_{\rm d})$-norm; i.e., there exists $C > 0$, independent of h, such that*

$$\|\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl},\Omega_{\rm d})} \le C \|\mathbf{curl}\, \boldsymbol{v}\|_{0,\Omega_{\rm d}} \qquad \forall \boldsymbol{v} \in V_{0,h}(\Omega_{\rm d}). \tag{3.34}$$

Proof. See, for instance, [45, Theorem 4.7]. □

We will also need the following result deduced from Proposition 3.3 of [11], which makes use of the quasi-uniformity of $\{\mathcal{T}_h\}_h$.

Lemma 3.5.2 *Let*

$$X_h(\Omega_{\rm c}) := \{\boldsymbol{v}|_{\Omega_{\rm c}} : \ \boldsymbol{v} \in X_h(\Omega)\} \quad \text{and} \quad X_h(\Omega_{\rm d}) := \{\boldsymbol{v}|_{\Omega_{\rm d}} : \ \boldsymbol{v} \in X_h(\Omega)\}.$$

There exists a linear operator

$$\mathcal{F}_h : \ \boldsymbol{\gamma}_\tau(X_h(\Omega_{\rm c})) \to X_h(\Omega_{\rm d})$$

such that $\boldsymbol{\gamma}_\tau(\mathcal{F}_h \boldsymbol{\eta}_h) = \boldsymbol{\eta}_h$ and

$$\|\mathcal{F}_h \boldsymbol{\eta}_h\|_{\mathbf{H}(\mathbf{curl},\Omega_{\rm d})} \le C \|\boldsymbol{\eta}_h\|_{\mathbf{H}^{-1/2}({\rm div}_\Gamma,\Gamma)} \quad \forall \boldsymbol{\eta}_h \in \boldsymbol{\gamma}_\tau(X_h(\Omega_{\rm c})),$$

for some positive constant C independent of h.

Lemma 3.5.3 *The linear mapping*

$$\begin{aligned}\mathcal{E}_h :\ X_h(\Omega_c) &\ \to\ V_{0,h}(\Omega)\\ \boldsymbol{v}_c &\ \mapsto\ \mathcal{E}_h\boldsymbol{v}_c\end{aligned}$$

characterized by $(\mathcal{E}_h\boldsymbol{v}_c)|_{\Omega_c} = \boldsymbol{v}_c$ *and*

$$(\mathbf{curl}\,\mathcal{E}_h\boldsymbol{v}_c, \mathbf{curl}\,\boldsymbol{w})_{0,\Omega_d} = 0 \qquad \forall \boldsymbol{w} \in V_{0,h}(\Omega_d) \tag{3.35}$$

is well defined and bounded uniformly in h*.*

Proof. Combining Theorem 3.2.1 and Lemma 3.5.2, we deduce that the linear mapping $\mathcal{L}_h :\ X_h(\Omega_c) \to X_h(\Omega_d)$ given by $\mathcal{L}_h\boldsymbol{v}_c = \mathcal{F}_h(\boldsymbol{\gamma}_\tau\boldsymbol{v}_c)$ is uniformly bounded, namely, there exists $C_0 > 0$, independent of h, such that

$$\|\mathcal{L}_h\boldsymbol{v}_c\|_{\mathbf{H}(\mathbf{curl},\Omega_d)} \le C_0\|\boldsymbol{v}_c\|_{\mathbf{H}(\mathbf{curl},\Omega_c)} \qquad \forall \boldsymbol{v}_c \in X_h(\Omega_c). \tag{3.36}$$

The mixed version of (3.35) consists of finding $\boldsymbol{z}_h \in \mathcal{L}_h\boldsymbol{v}_c + X_{0,h}(\Omega_d)$ and $\rho_h \in M_h(\Omega_d)$ such that

$$\begin{aligned}(\mathbf{curl}\,\boldsymbol{z}_h, \mathbf{curl}\,\boldsymbol{w})_{0,\Omega_d} + b(\boldsymbol{w}, \rho_h) &= 0 \qquad \forall \boldsymbol{w} \in X_{0,h}(\Omega_d),\\ b(\boldsymbol{z}_h, \vartheta) &= 0 \qquad \forall \vartheta \in M_h(\Omega_d),\end{aligned}$$

where $X_{0,h}(\Omega_d) := X_h(\Omega_d) \cap \mathbf{H}_0(\mathbf{curl}, \Omega_d)$. Similarly to the continuous case,

$$\mathbf{grad}(M_h(\Omega_d)) \subset X_{0,h}(\Omega_d)$$

and hence

$$\sup_{\boldsymbol{z}\in X_{0,h}(\Omega_d)} \frac{b(\boldsymbol{z}, \vartheta)}{\|\boldsymbol{z}\|_{\mathbf{H}(\mathbf{curl},\Omega_d)}} \ge \varepsilon_0 \frac{(\mathbf{grad}\,\vartheta, \mathbf{grad}\,\vartheta)_{0,\Omega_d}}{\|\,\mathbf{grad}\,\vartheta\|_{\mathbf{H}(\mathbf{curl},\Omega_d)}} = \varepsilon_0|\vartheta|_{1,\Omega_d} \qquad \forall \vartheta \in M_h(\Omega_d).$$

This discrete inf-sup condition and (3.34) allow us to apply again the Babuška-Brezzi theory to deduce that $\boldsymbol{z}_h$ is well-defined and

$$\|\boldsymbol{z}_h\|_{\mathbf{H}(\mathbf{curl},\Omega_d)} \le C_1\|\mathcal{L}_h\boldsymbol{v}_c\|_{\mathbf{H}(\mathbf{curl},\Omega_c)},$$

with $C_1 > 0$ independent of h. If we define $\mathcal{E}_h\boldsymbol{v}_c := \boldsymbol{z}_h$, clearly $(\mathcal{E}_h\boldsymbol{v}_c)|_{\Omega_c} = \boldsymbol{v}_c$ and (3.35) holds true. Moreover, these two conditions clearly determine $\mathcal{E}_h\boldsymbol{v}_c$ uniquely and applying (3.36) we have

$$\|\mathcal{E}_h\boldsymbol{v}_c\|_{\mathbf{H}(\mathbf{curl};\Omega)} \le \sqrt{1 + (C_0C_1)^2}\|\boldsymbol{v}_c\|_{\mathbf{H}(\mathbf{curl},\Omega_c)} \qquad \forall \boldsymbol{v}_c \in \mathbf{H}(\mathbf{curl};\Omega),$$

from which the result follows. □

Proceeding exactly as in the previous section we obtain the following result.

Lemma 3.5.4 *The bilinear form* $(\cdot,\cdot)_{V_0(\Omega)}$ *induces a norm on* $V_{0,h}(\Omega)$ *uniformly equivalent to the* $\mathbf{H}(\mathbf{curl};\Omega)$*-norm; i.e., there exists* $C_1 > 0$ *and* $C_2 > 0$*, independent of* h*, such that*

$$C_1\|\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)} \leq \|\boldsymbol{v}\|_{V_0(\Omega)} \leq C_2\|\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)} \qquad \forall \boldsymbol{v} \in V_{0,h}(\Omega). \tag{3.37}$$

Moreover, we have the following $(\cdot,\cdot)_{V_0(\Omega)}$*-orthogonal decomposition:*

$$V_{0,h}(\Omega) = \widetilde{V_{0,h}(\Omega_{\rm d})} \oplus \mathcal{E}_h(X_h(\Omega_{\rm c})), \tag{3.38}$$

where $\widetilde{V_{0,h}(\Omega_{\rm d})}$ *is the subspace of* $V_{0,h}(\Omega)$ *obtained by extending the functions of* $V_{0,h}(\Omega_{\rm d})$ *by zero to the whole domain* Ω*.*

Theorem 3.5.1 *Problem* (3.33) *has a unique solution* $(\boldsymbol{u}_h, \lambda_h)$ *with an identically vanishing discrete Lagrange multiplier* λ_h*.*

Proof. According to (3.38), we look for a solution of problem (3.33) written as follows: $\boldsymbol{u}_h = \boldsymbol{u}_{{\rm d},h} + \mathcal{E}_h(\boldsymbol{u}_{{\rm c},h})$, with $\boldsymbol{u}_{{\rm d},h}(t) \in \widetilde{V_{0,h}(\Omega_{\rm d})}$ and $\boldsymbol{u}_{{\rm c},h} \in X_h(\Omega_{\rm c})$. Notice that $\boldsymbol{u}_{{\rm d},h}(t)|_{\Omega_{\rm d}} \in V_{0,h}(\Omega_{\rm d})$ must be the unique solution of the problem

$$\left(\frac{1}{\mu(t)}\,\mathbf{curl}\,\boldsymbol{u}_{{\rm d},h}(t), \mathbf{curl}\,\boldsymbol{v}\right)_{0,\Omega_{\rm d}} = (\boldsymbol{f}(t),\boldsymbol{v})_{0,\Omega_{\rm d}} \qquad \forall \boldsymbol{v} \in V_{0,h}(\Omega_{\rm d}).$$

The other term $\boldsymbol{u}_{{\rm c},h}$ has to be the unique solution of the finite dimensional initial value problem

$$\frac{d}{dt}(\boldsymbol{u}_{{\rm c},h}(t),\boldsymbol{v})_\sigma + a(t;\mathcal{E}_h\boldsymbol{u}_{{\rm c},h}(t),\mathcal{E}_h\boldsymbol{v}) = (\boldsymbol{f}(t),\mathcal{E}_h\boldsymbol{v})_{0,\Omega} \qquad \forall \boldsymbol{v} \in X_h(\Omega_{\rm c}),$$
$$\boldsymbol{u}_{{\rm c},h}(0) = \mathbf{0}.$$

It only remains to prove the existence and uniqueness of the Lagrange multiplier λ_h. With this aim we notice that the functional defined by

$$\langle \mathcal{G}_h(t),\boldsymbol{v}\rangle := \int_0^t \Big[(\boldsymbol{f}(s),\boldsymbol{v})_{0,\Omega} - a(s;\boldsymbol{u}_h(s),\boldsymbol{v})\Big]\,ds - (\boldsymbol{u}_h(t),\boldsymbol{v})_\sigma$$

vanishes on the discrete kernel:

$$\langle \mathcal{G}_h(t),\boldsymbol{v}\rangle = 0 \qquad \forall \boldsymbol{v} \in V_{0,h}(\Omega).$$

Hence, the discrete inf-sup condition,

$$\sup_{\boldsymbol{v}\in X_h(\Omega)} \frac{b(\boldsymbol{v},\vartheta)}{\|\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)}} \geq \frac{b(\widetilde{\mathbf{grad}}\,\vartheta,\vartheta)}{\|\widetilde{\mathbf{grad}}\,\vartheta\|_{\mathbf{H}(\mathbf{curl};\Omega)}} = \varepsilon_0|\vartheta|_{1,\Omega_{\rm d}} \qquad \forall \vartheta \in M_h(\Omega_{\rm d}), \tag{3.39}$$

implies that there exists a unique $\lambda_h(t)$ satisfying

$$b(\boldsymbol{v}, \lambda_h(t)) = \langle \mathcal{G}_h(t), \boldsymbol{v} \rangle \qquad \forall \boldsymbol{v} \in X_h(\Omega).$$

By differentiating the last equation we obtain that $\lambda_h(t)$ solves (3.33).

Finally, since $\mathbf{grad}(M_h(\Omega_{\rm d})) \subset X_{0,h}(\Omega_{\rm d})$, we are allowed to test the first equation of (3.33) with $\mathbf{grad}\,\lambda_h(t)$ extended by zero to the whole Ω to obtain

$$\frac{d}{dt} b(\widetilde{\mathbf{grad}}\,\lambda_h(t), \lambda_h(t)) = (\boldsymbol{f}(t), \mathbf{grad}\,\lambda_h(t))_{0,\Omega_{\rm d}} = 0.$$

Therefore,

$$\varepsilon_0 |\lambda_h(t)|^2_{1,\Omega_{\rm d}} = b(\widetilde{\mathbf{grad}}\,\lambda_h(t), \lambda_h(t)) = \langle \mathcal{G}_h(0), \widetilde{\mathbf{grad}}\,\lambda_h(0) \rangle = 0$$

and the result follows. □

3.5.1 Error estimates

Our next goal is to prove error estimates for our semi-discrete scheme. Notice that as $\lambda = \lambda_h = 0$, we will only be concerned with error estimates for the main variable $\boldsymbol{u}$.

Consider the linear projection operator $\Pi_h : \mathbf{H}_0(\mathbf{curl}, \Omega) \to V_{0,h}(\Omega)$ defined by

$$\Pi_h \boldsymbol{v} \in V_{0,h}(\Omega) : \qquad (\Pi_h \boldsymbol{v}, \boldsymbol{z})_{\mathbf{H}(\mathbf{curl};\Omega)} = (\boldsymbol{v}, \boldsymbol{z})_{\mathbf{H}(\mathbf{curl};\Omega)} \qquad \forall \boldsymbol{z} \in V_{0,h}(\Omega). \tag{3.40}$$

Lemma 3.5.5 *There exists $C > 0$, independent of h, such that*

$$\|\boldsymbol{v} - \Pi_h \boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)} \leq C \inf_{\boldsymbol{z} \in X_h(\Omega)} \|\boldsymbol{v} - \boldsymbol{z}\|_{\mathbf{H}(\mathbf{curl};\Omega)} \tag{3.41}$$

for all $\boldsymbol{v} \in V_0(\Omega)$.

Proof. From the definition of Π_h we deduce that

$$\|\boldsymbol{v} - \Pi_h \boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)} \leq \inf_{\boldsymbol{z} \in V_{0,h}(\Omega)} \|\boldsymbol{v} - \boldsymbol{z}\|_{\mathbf{H}(\mathbf{curl};\Omega)}.$$

Furthermore, since b satisfies the discrete inf-sup condition (cf. the proof of Lemma 3.5.3) and $\boldsymbol{v} \in V_0(\Omega)$, we can use the trick given in [43, Theorem II-1.1] to estimate the right hand side of the previous inequality. In fact, let $\boldsymbol{y} \in X_h(\Omega)$. The discrete inf-sup condition implies that there exists a unique

$$\boldsymbol{w} \in V_{0,h}(\Omega)^{\perp} := \left\{ \boldsymbol{w} \in X_h(\Omega) : (\boldsymbol{w}, \boldsymbol{v})_{\mathbf{H}(\mathbf{curl};\Omega)} = 0 \quad \forall \boldsymbol{v} \in V_{0,h}(\Omega) \right\}$$

such that

$$b(\boldsymbol{w}, q) = b(\boldsymbol{v} - \boldsymbol{y}, q) \qquad \forall q \in M_h(\Omega_{\mathrm{d}}),$$

with

$$\|\boldsymbol{w}\|_{\mathbf{H}(\mathbf{curl};\Omega)} \leq \frac{1}{\varepsilon_0} \sup_{q \in M_h(\Omega_{\mathrm{d}})} \frac{b(\boldsymbol{v} - \boldsymbol{y}, q)}{|q|_{1,\Omega_{\mathrm{d}}}} \leq \frac{1}{\varepsilon_0} \|\boldsymbol{v} - \boldsymbol{y}\|_{\mathbf{H}(\mathbf{curl};\Omega)}.$$

Thus, if we define $\tilde{\boldsymbol{z}} := \boldsymbol{w} + \boldsymbol{y}$, then $\tilde{\boldsymbol{z}} \in V_{0,h}(\Omega)$. Consequently,

$$\begin{aligned} \inf_{\boldsymbol{z} \in V_{0,h}(\Omega)} \|\boldsymbol{v} - \boldsymbol{z}\|_{\mathbf{H}(\mathbf{curl};\Omega)} &\leq \|\boldsymbol{v} - \tilde{\boldsymbol{z}}\|_{\mathbf{H}(\mathbf{curl};\Omega)} \\ &\leq \|\boldsymbol{v} - \boldsymbol{y}\|_{\mathbf{H}(\mathbf{curl};\Omega)} + \|\boldsymbol{w}\|_{\mathbf{H}(\mathbf{curl};\Omega)} \\ &\leq \left(1 + \frac{1}{\varepsilon_0}\right) \|\boldsymbol{v} - \boldsymbol{y}\|_{\mathbf{H}(\mathbf{curl};\Omega)}. \end{aligned}$$

As $\boldsymbol{y} \in X_h(\Omega)$ is arbitrary, combining this last inequality with (3.42), we conclude that

$$\inf_{\boldsymbol{z} \in V_{0,h}(\Omega)} \|\boldsymbol{v} - \boldsymbol{z}\|_{\mathbf{H}(\mathbf{curl};\Omega)} \leq C \inf_{\boldsymbol{z} \in X_h(\Omega)} \|\boldsymbol{v} - \boldsymbol{z}\|_{\mathbf{H}(\mathbf{curl};\Omega)},$$

which proves (3.41). □

In order to obtain the error estimates, from now on we assume that for almost every $\boldsymbol{x} \in \Omega$, $\mu(\boldsymbol{x}, t)$ is differentiable with respect to t and that there exists a constant $\widetilde{\mu}_1 > 0$ such that

$$|\partial_t \mu(\boldsymbol{x}, t)| \leq \widetilde{\mu}_1 \qquad \forall t \in (0, T), \quad \text{a.e. } \boldsymbol{x} \in \Omega.$$

Lemma 3.5.6 *Let $\boldsymbol{\rho}_h(t) := \boldsymbol{u}(t) - \Pi_h \boldsymbol{u}(t)$ and $\boldsymbol{\delta}_h(t) := \Pi_h \boldsymbol{u}(t) - \boldsymbol{u}_h(t)$. There exists a constant $C > 0$, independent of h, such that*

$$\begin{aligned} &\sup_{t \in [0,T]} \|\boldsymbol{\delta}_h(t)\|_\sigma^2 + \sup_{t \in [0,T]} \|\mathbf{curl}\, \boldsymbol{\delta}_h(t)\|_{0,\Omega}^2 \\ &+ \int_0^T \|\mathbf{curl}\, \boldsymbol{\delta}_h(t)\|_{0,\Omega}^2 \, dt + \int_0^T \|\partial_t \boldsymbol{\delta}_h(t)\|_\sigma^2 \, dt \\ &\leq C \left\{ \int_0^T \|\partial_t \boldsymbol{\rho}_h(t)\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2 \, dt + \sup_{t \in (0,T)} \|\boldsymbol{\rho}_h(t)\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2 \right\}. \end{aligned} \tag{3.42}$$

Proof. A straightforward computation yields

$$(\partial_t \boldsymbol{\delta}_h(t), \boldsymbol{v})_\sigma + a(t; \boldsymbol{\delta}_h(t), \boldsymbol{v}) = -(\partial_t \boldsymbol{\rho}_h(t), \boldsymbol{v})_\sigma - a(t; \boldsymbol{\rho}_h(t), \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in V_{0,h}(\Omega). \tag{3.43}$$

By taking $\boldsymbol{v} = \boldsymbol{\delta}_h(t)$ in the last identity and using the Cauchy-Schwartz inequality together with (3.9), we obtain

$$\frac{d}{dt} \|\boldsymbol{\delta}_h(t)\|_\sigma^2 + \mu_1^{-1} \|\mathbf{curl}\, \boldsymbol{\delta}_h(t)\|_{0,\Omega}^2 \leq \|\boldsymbol{\delta}_h(t)\|_\sigma^2 + \|\partial_t \boldsymbol{\rho}_h(t)\|_\sigma^2 + \frac{\mu_1}{\mu_0^2} \|\mathbf{curl}\, \boldsymbol{\rho}_h(t)\|_{0,\Omega}^2.$$

We now integrate over $[0,t]$ (note that $\boldsymbol{\delta}_h(0)=\mathbf{0}$) and use Gronwall's inequality to obtain

$$\|\boldsymbol{\delta}_h(t)\|_\sigma^2+\mu_1^{-1}\int_0^t \|\operatorname{\mathbf{curl}}\boldsymbol{\delta}_h(s)\|_{0,\Omega}^2\,ds\le C_1\int_0^T \left[\|\partial_t\boldsymbol{\rho}_h(t)\|_\sigma^2+\|\boldsymbol{\rho}_h(t)\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2\right]dt. \quad (3.44)$$

Analogously, taking $\boldsymbol{v}=\partial_t\boldsymbol{\delta}_h(t)$ in (3.43) and using the identity

$$a(t;\boldsymbol{z},\partial_t\boldsymbol{w})=\frac{d}{dt}a(t;\boldsymbol{z},\boldsymbol{w})-\int_\Omega\frac{1}{\mu(t)}\operatorname{\mathbf{curl}}\partial_t\boldsymbol{z}\cdot\operatorname{\mathbf{curl}}\boldsymbol{w}+\int_\Omega\frac{\partial_t\mu(t)}{\mu(t)^2}\operatorname{\mathbf{curl}}\boldsymbol{z}\cdot\operatorname{\mathbf{curl}}\boldsymbol{w},$$

we obtain

$$\begin{aligned}&\|\partial_t\boldsymbol{\delta}_h(t)\|_\sigma^2+\frac{1}{2}\frac{d}{dt}a(t;\boldsymbol{\delta}_h(t),\boldsymbol{\delta}_h(t))+\frac{1}{2}\int_\Omega\frac{\partial_t\mu(t)}{\mu(t)^2}|\operatorname{\mathbf{curl}}\boldsymbol{\delta}_h(t)|^2\\ &\quad=-(\partial_t\boldsymbol{\rho}_h(t),\partial_t\boldsymbol{\delta}_h(t))_\sigma-\frac{d}{dt}a(t;\boldsymbol{\rho}_h(t),\boldsymbol{\delta}_h(t))-\int_\Omega\frac{1}{\mu(t)}\operatorname{\mathbf{curl}}\partial_t\boldsymbol{\rho}_h(t)\cdot\operatorname{\mathbf{curl}}\boldsymbol{\delta}_h(t)\\ &\qquad+\int_\Omega\frac{\partial_t\mu(t)}{\mu(t)^2}\operatorname{\mathbf{curl}}\boldsymbol{\rho}_h(t)\cdot\operatorname{\mathbf{curl}}\boldsymbol{\delta}_h(t).\end{aligned}$$

Integrating over $[0,t]$ and using the Cauchy-Schwartz inequality lead to

$$\begin{aligned}&\int_0^t\|\partial_t\boldsymbol{\delta}_h(s)\|_\sigma^2\,ds+\|\operatorname{\mathbf{curl}}\boldsymbol{\delta}_h(t)\|_{0,\Omega}^2\\ &\qquad\le C_2\left\{\int_0^T\|\partial_t\boldsymbol{\rho}_h(t)\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2\,dt+\sup_{t\in[0,T]}\|\operatorname{\mathbf{curl}}\boldsymbol{\rho}_h(t)\|_{0,\Omega}^2\right.\\ &\qquad\qquad\left.+\int_0^t\|\operatorname{\mathbf{curl}}\boldsymbol{\delta}_h(s)\|_{0,\Omega}^2\,ds\right\}.\end{aligned}$$

Finally, using Gronwall's Lemma, we have

$$\begin{aligned}&\int_0^t\|\partial_t\boldsymbol{\delta}_h(s)\|_\sigma^2\,ds+\|\operatorname{\mathbf{curl}}\boldsymbol{\delta}_h(t)\|_{0,\Omega}^2\\ &\qquad\le C_3\left\{\int_0^T\|\partial_t\boldsymbol{\rho}_h(t)\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2\,dt+\sup_{t\in[0,T]}\|\operatorname{\mathbf{curl}}\boldsymbol{\rho}_h(t)\|_{0,\Omega}^2\right\}.\end{aligned}$$

The last inequality and (3.44) yield (3.42). □

Theorem 3.5.2 *Assume that* $\boldsymbol{u}\in\mathrm{H}^1(0,T;\mathbf{H}(\mathbf{curl};\Omega))$ *and let* $\boldsymbol{e}_h(t):=\boldsymbol{u}(t)-\boldsymbol{u}_h(t)$. *There exists* $C>0$, *independent of* h, *such that*

$$\begin{aligned}&\sup_{t\in[0,T]}\|\boldsymbol{e}_h(t)\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2+\int_0^T\|\boldsymbol{e}_h(t)\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2\,dt+\int_0^T\|\partial_t\boldsymbol{e}_h(t)\|_\sigma^2\,dt\\ &\quad\le C\left\{\int_0^T\left[\inf_{\boldsymbol{v}\in X_h(\Omega)}\|\boldsymbol{u}(t)-\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2+\inf_{\boldsymbol{v}\in X_h(\Omega)}\|\partial_t\boldsymbol{u}(t)-\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2\right]dt\right.\\ &\qquad\left.+\sup_{t\in[0,T]}\inf_{\boldsymbol{v}\in X_h(\Omega)}\|\boldsymbol{u}(t)-\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2\right\}.\end{aligned}$$

Proof. Notice that the regularity assumption on $\boldsymbol{u}$ allows us to commute the time derivative and Π_h:

$$\partial_t \left(\Pi_h \boldsymbol{u}(t)\right) = \Pi_h(\partial_t \boldsymbol{u}(t)).$$

Hence, Lemma 3.5.5 implies that

$$\|\boldsymbol{\rho}_h(t)\|_{\mathbf{H}(\mathbf{curl};\Omega)} \leq C \inf_{\boldsymbol{v}\in X_h(\Omega)} \|\boldsymbol{u}(t) - \boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)} \tag{3.45}$$

and

$$\|\partial_t\boldsymbol{\rho}_h(t)\|_{\mathbf{H}(\mathbf{curl};\Omega)} \leq C \inf_{\boldsymbol{v}\in X_h(\Omega)} \|\partial_t\boldsymbol{u}(t) - \boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)}. \tag{3.46}$$

Thus, the result follows by writing $\boldsymbol{e}_h(t) = \boldsymbol{\rho}_h(t) + \boldsymbol{\delta}_h(t)$ and using the estimates for $\boldsymbol{\delta}_h(t)$ from Lemma 3.5.6. □

For any $r \geq 0$, we consider the Sobolev space

$$\mathbf{H}^r(\mathbf{curl}, \Omega_c) := \left\{\boldsymbol{v} \in \mathrm{H}^r(\Omega_c)^3 : \ \mathbf{curl}\, \boldsymbol{v} \in \mathrm{H}^r(\Omega_c)^3\right\}$$

endowed with the norm $\|\boldsymbol{v}\|^2_{\mathbf{H}^r(\mathbf{curl},\Omega_c)} := \|\boldsymbol{v}\|^2_{r,\Omega_c} + \|\mathbf{curl}\,\boldsymbol{v}\|^2_{r,\Omega_c}$ and analogously for Ω_d. It is well known that the Nédélec operator interpolation $\mathcal{I}_h\boldsymbol{v} \in X_h(\Omega_c)$ (and $\mathcal{I}_h\boldsymbol{w} \in X_h(\Omega_d)$) is well defined for any $\boldsymbol{v} \in \mathbf{H}^r(\mathbf{curl}, \Omega_c)$ (resp. $\boldsymbol{w} \in \mathbf{H}^r(\mathbf{curl}, \Omega_d)$) with $r > 1/2$; see, for instance, Lemma 5.1 of [11] or Lemma 4.7 of [14]. Moreover, if we introduce the space

$$\boldsymbol{\mathcal{X}} := \{\boldsymbol{v} \in \mathbf{H}(\mathbf{curl}; \Omega) : \ \exists r > 1/2 \quad \boldsymbol{v}|_{\Omega_c} \in \mathbf{H}^r(\mathbf{curl}, \Omega_c) \text{ and } \boldsymbol{v}|_{\Omega_d} \in \mathbf{H}^r(\mathbf{curl}, \Omega_d)\} \tag{3.47}$$

endowed with the $\mathbf{H}^r(\mathbf{curl}, \Omega_c)\times\mathbf{H}^r(\mathbf{curl}, \Omega_d)$-norm, then $\mathcal{I}_h : \boldsymbol{\mathcal{X}} \to X_h(\Omega)$ is well defined, bounded uniformly in h and the following interpolation error estimate holds true (see Lemma 5.1 of [16] or Proposition 5.6 of [11]):

$$\|\boldsymbol{v} - \mathcal{I}_h\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)} \leq Ch^{\min\{r,m\}}\|\boldsymbol{v}\|_{\boldsymbol{\mathcal{X}}} \qquad \forall \boldsymbol{v} \in \mathbf{H}^r(\mathbf{curl}, \Omega). \tag{3.48}$$

Corollary 3.5.1 *If* $\boldsymbol{u} \in \mathrm{H}^1(0, T; \boldsymbol{\mathcal{X}} \cap \mathbf{H}_0(\mathbf{curl}, \Omega))$*, then*

$$\sup_{t\in[0,T]} \|\boldsymbol{e}_h(t)\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} + \int_0^T \|\boldsymbol{e}_h(t)\|^2_{\mathbf{H}(\mathbf{curl};\Omega)}\, dt + \int_0^T \|\partial_t\boldsymbol{e}_h(t)\|^2_\sigma\, dt \leq Ch^{2l}\left\{\sup_{t\in[0,T]} \|\boldsymbol{u}(t)\|^2_{\boldsymbol{\mathcal{X}}} + \int_0^T \|\partial_t\boldsymbol{u}(t)\|^2_{\boldsymbol{\mathcal{X}}}\, dt\right\},$$

with $l := \min\{r, m\}$.

Proof. It is a direct consequence of Theorem 3.5.2 and the interpolation error estimate (3.48). □

Remark 3.5.1 *The eddy currents*

$$\sigma \boldsymbol{E}(\boldsymbol{x}, t) = \sigma \partial_t \boldsymbol{u}(\boldsymbol{x}, t)$$

can be approximated by $\sigma \boldsymbol{E}_h(\boldsymbol{x}, t)$*, where* $\boldsymbol{E}_h(\boldsymbol{x}, t) := \partial_t \boldsymbol{u}_h(\boldsymbol{x}, t)$*. In fact, Theorem 3.5.2 and Corollary 3.5.1 provide convergence estimates for* $\sigma \boldsymbol{E} - \sigma \boldsymbol{E}_h$ *in the* $\mathrm{L}^2(0, T; \mathrm{L}^2(\Omega_{\mathrm{c}}))$*-norm. On the other hand, by virtue of* (3.11)*, Theorem 3.5.2 and Corollary 3.5.1 also yield estimates for the approximation of the magnetic induction* $\boldsymbol{B} := \mu \boldsymbol{H}$*.*

3.6 Analysis of a fully-discrete scheme.

We consider a uniform partition $\{t_n := n\Delta t : \; n = 0, \ldots, N\}$ of $[0, T]$ with a step size $\Delta t := \frac{T}{N}$. For any finite sequence $\{\theta^n : \; n = 0, \cdots, N\}$, let

$$\bar{\partial}\theta^n := \frac{\theta^n - \theta^{n-1}}{\Delta t}, \qquad n = 1, 2, \ldots, N.$$

The fully-discrete version of problem (3.16) reads as follows:
Find $(\boldsymbol{u}_h^n, \lambda_h^n) \in X_h(\Omega) \times M_h(\Omega_{\mathrm{d}}), \; n = 1, \cdots, N$, such that

$$\begin{aligned} (\bar{\partial}\boldsymbol{u}_h^n, \boldsymbol{v})_\sigma + b(\boldsymbol{v}, \bar{\partial}\lambda_h^n) + a(t_n; \boldsymbol{u}_h^n, \boldsymbol{v}) &= (\boldsymbol{f}(t_n), \boldsymbol{v})_{0,\Omega} \qquad \forall \boldsymbol{v} \in X_h(\Omega), \\ b(\boldsymbol{u}_h^n, \mu) &= 0 \qquad \forall \mu \in M_h(\Omega_{\mathrm{d}}), \\ \boldsymbol{u}_h^0|_{\Omega_{\mathrm{c}}} &= \boldsymbol{0}, \\ \lambda_h^0 &= 0. \end{aligned} \tag{3.49}$$

Hence, at each iteration step we have to find $(\boldsymbol{u}_h^n, \lambda_h^n) \in X_h(\Omega) \times M_h(\Omega_{\mathrm{d}})$ such that

$$\begin{aligned} (\boldsymbol{u}_h^n, \boldsymbol{v})_\sigma + \Delta t \, a(t_n; \boldsymbol{u}_h^n, \boldsymbol{v}) + b(\boldsymbol{v}, \lambda_h^n) &= F_n(\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in X_h(\Omega), \\ b(\boldsymbol{u}_h^n, \mu) &= 0 \qquad \forall \mu \in M_h(\Omega_{\mathrm{d}}), \end{aligned}$$

where

$$F_n(\boldsymbol{v}) := \Delta t(\boldsymbol{f}(t_n), \boldsymbol{v})_{0,\Omega} + (\boldsymbol{u}_h^{n-1}, \boldsymbol{v})_\sigma + b(\boldsymbol{v}, \lambda_h^{n-1}).$$

The existence and uniqueness of $(\boldsymbol{u}_h^n, \lambda_h^n)$ is a direct consequence of the Babuška-Brezzi theory. Indeed, as shown in the proof of Theorem 3.5.1, the bilinear form b satisfies the discrete inf-sup condition and $\mathcal{A}(\boldsymbol{v},\boldsymbol{w}) := (\boldsymbol{v},\boldsymbol{w})_\sigma + \Delta t\, a(t_n;\boldsymbol{v},\boldsymbol{w})$ induces a norm on its kernel $V_{0,h}(\Omega)$ (cf. Lemma 3.5.4). Furthermore, testing the first equation of (3.49) with $\widetilde{\mathbf{grad}}\,\lambda_h^n$ and taking into account (3.29) leads to

$$\varepsilon_0|\lambda_h^n|^2_{1,\Omega_\mathrm{d}} = b(\mathbf{grad}\,\lambda_h^n,\lambda_h^n) = b(\mathbf{grad}\,\lambda_h^n,\lambda_h^{n-1}), \qquad n=1,\dots,N.$$

Consequently, the condition $\lambda_h^0=0$ implies that

$$\lambda_h^n = 0, \qquad n=1,\dots,N.$$

3.6.1 Error estimates

Lemma 3.6.1 *Let $\boldsymbol{\rho}^n := \boldsymbol{u}(t_n)-\Pi_h\boldsymbol{u}(t_n)$, $\boldsymbol{\delta}^n := \Pi_h\boldsymbol{u}(t_n)-\boldsymbol{u}_h^n$ and $\boldsymbol{\tau}^n := \bar\partial\boldsymbol{u}(t_n)-\partial_t\boldsymbol{u}(t_n)$. There exists a constant $C>0$, independent of h and Δt, such that*

$$\begin{aligned}\|\boldsymbol{\delta}^n\|_\sigma^2 + \|\mathbf{curl}\,\boldsymbol{\delta}^n\|^2_{0,\Omega} + \Delta t\sum_{k=1}^n\|\mathbf{curl}\,\boldsymbol{\delta}^k\|^2_{0,\Omega} + \Delta t\sum_{k=1}^n\|\bar\partial\boldsymbol{\delta}^k\|^2_\sigma&\\ \le C\Delta t\left(\sum_{k=1}^N\|\bar\partial\boldsymbol{\rho}^k\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} + \sum_{k=1}^N\|\mathbf{curl}\,\boldsymbol{\rho}^k\|^2_{0,\Omega} + \sum_{k=1}^N\|\boldsymbol{\tau}^k\|^2_\sigma\right)&,\end{aligned}\tag{3.50}$$

for all $n=1,\dots,N$.

Proof. It is straightforward to show that

$$(\bar\partial\boldsymbol{\delta}^k,\boldsymbol{v})_\sigma + a(t_k;\boldsymbol{\delta}^k,\boldsymbol{v}) = -(\bar\partial\boldsymbol{\rho}^k,\boldsymbol{v})_\sigma - a(t_k;\boldsymbol{\rho}^k,\boldsymbol{v}) + (\boldsymbol{\tau}^k,\boldsymbol{v})_\sigma \qquad \forall\boldsymbol{v}\in V_{0,h}.\tag{3.51}$$

Choosing $\boldsymbol{v}=\boldsymbol{\delta}^k$ in the last identity and using the estimates

$$a(t_k;\boldsymbol{\delta}^k,\boldsymbol{\delta}^k) \ge \mu_1^{-1}\|\mathbf{curl}\,\boldsymbol{\delta}^k\|^2_{0,\Omega} \quad\text{and}\quad (\bar\partial\boldsymbol{\delta}^k,\boldsymbol{\delta}^k)_\sigma \ge \frac{1}{2\Delta t}\left(\|\boldsymbol{\delta}^k\|^2_\sigma - \|\boldsymbol{\delta}^{k-1}\|^2_\sigma\right),$$

together with the Cauchy-Schwartz inequality, yield

$$\begin{aligned}\|\boldsymbol{\delta}^k\|^2_\sigma - \|\boldsymbol{\delta}^{k-1}\|^2_\sigma + \Delta t\,\mu_1^{-1}\|\mathbf{curl}\,\boldsymbol{\delta}^k\|^2_{0,\Omega}&\\ \le \frac{\Delta t}{2T}\|\boldsymbol{\delta}^k\|^2_\sigma + C_1\Delta t\left(\|\bar\partial\boldsymbol{\rho}^k\|^2_\sigma + \|\mathbf{curl}\,\boldsymbol{\rho}^k\|_{0,\Omega} + \|\boldsymbol{\tau}^k\|^2_\sigma\right)&.\end{aligned}\tag{3.52}$$

In particular,

$$\|\boldsymbol{\delta}^k\|^2_\sigma - \|\boldsymbol{\delta}^{k-1}\|^2_\sigma \le \frac{\Delta t}{2T}\|\boldsymbol{\delta}^k\|^2_\sigma + C_1\Delta t\left(\|\bar\partial\boldsymbol{\rho}^k\|^2_\sigma + \|\mathbf{curl}\,\boldsymbol{\rho}^k\|_{0,\Omega} + \|\boldsymbol{\tau}^k\|^2_\sigma\right).$$

Then, summing over k and using the discrete Gronwall's Lemma (see, for instance, [62, Lemma 1.4.2]) lead to

$$\|\boldsymbol{\delta}^n\|_\sigma^2 \leq C_2 \Delta t \sum_{k=1}^{n} \left(\|\bar{\partial}\boldsymbol{\rho}^k\|_\sigma^2 + \|\,\mathbf{curl}\,\boldsymbol{\rho}^k\|_{0,\Omega}^2 + \|\boldsymbol{\tau}^k\|_\sigma^2 \right),$$

for $n = 1, \ldots, N$. Inserting the last inequality in (3.52) and summing over k we have the estimate

$$\begin{aligned} &\|\boldsymbol{\delta}^n\|_\sigma^2 + \Delta t \sum_{k=1}^{n} \|\,\mathbf{curl}\,\boldsymbol{\delta}^k\|_{0,\Omega}^2 \\ &\qquad \leq C_3 \Delta t \left(\sum_{k=1}^{n} \|\bar{\partial}\boldsymbol{\rho}^k\|_\sigma^2 + \sum_{k=1}^{n} \|\,\mathbf{curl}\,\boldsymbol{\rho}^k\|_{0,\Omega}^2 + \sum_{k=1}^{n} \|\boldsymbol{\tau}^k\|_\sigma^2 \right). \end{aligned} \tag{3.53}$$

Let us now take $\boldsymbol{v} = \bar{\partial}\boldsymbol{\delta}^k$ in (3.51):

$$\|\bar{\partial}\boldsymbol{\delta}^k\|_\sigma^2 + a(t_k; \boldsymbol{\delta}^k, \bar{\partial}\boldsymbol{\delta}^k) = -(\bar{\partial}\boldsymbol{\rho}^k, \bar{\partial}\boldsymbol{\delta}^k)_\sigma - a(t_k; \boldsymbol{\rho}^k, \bar{\partial}\boldsymbol{\delta}^k) + (\boldsymbol{\tau}^k, \bar{\partial}\boldsymbol{\delta}^k)_\sigma. \tag{3.54}$$

Since the bilinear form $a(t_k; \cdot, \cdot)$ is nonnegative, we have that

$$\begin{aligned} a(t_k; \boldsymbol{\delta}^k, \bar{\partial}\boldsymbol{\delta}^k) &\geq \frac{1}{2\Delta t} \left[a(t_k; \boldsymbol{\delta}^k, \boldsymbol{\delta}^k) - a(t_k; \boldsymbol{\delta}^{k-1}, \boldsymbol{\delta}^{k-1}) \right] \\ &= \frac{1}{2\Delta t} \left[a(t_k; \boldsymbol{\delta}^k, \boldsymbol{\delta}^k) - a(t_{k-1}; \boldsymbol{\delta}^{k-1}, \boldsymbol{\delta}^{k-1}) \right] \\ &\quad + \frac{1}{2\Delta t} \left[a(t_{k-1}; \boldsymbol{\delta}^{k-1}, \boldsymbol{\delta}^{k-1}) - a(t_k; \boldsymbol{\delta}^{k-1}, \boldsymbol{\delta}^{k-1}) \right]. \end{aligned}$$

Then, there exists $\xi_k \in (t_{k-1}, t_k)$ such that

$$\begin{aligned} a(t_k; \boldsymbol{\delta}^k, \bar{\partial}\boldsymbol{\delta}^k) &\geq \frac{1}{2\Delta t} \left[a(t_k; \boldsymbol{\delta}^k, \boldsymbol{\delta}^k) - a(t_{k-1}; \boldsymbol{\delta}^{k-1}, \boldsymbol{\delta}^{k-1}) \right] \\ &\quad + \frac{1}{2} \int_\Omega \frac{\mu'(\xi_k)}{\mu(\xi_k)^2} |\,\mathbf{curl}\,\boldsymbol{\delta}^{k-1}|^2. \end{aligned} \tag{3.55}$$

On the other hand, a straightforward computation shows that

$$\begin{aligned} a(t_k; \boldsymbol{\rho}^k, \bar{\partial}\boldsymbol{\delta}^k) &= \frac{1}{\Delta t} \left[a(t_k; \boldsymbol{\rho}^k, \boldsymbol{\delta}^k) - a(t_{k-1}; \boldsymbol{\rho}^{k-1}, \boldsymbol{\delta}^{k-1}) \right] - a(t_k; \bar{\partial}\boldsymbol{\rho}^k, \boldsymbol{\delta}^{k-1}) \\ &\quad + \frac{1}{2} \int_\Omega \frac{\mu'(\xi_k)}{\mu(\xi_k)^2} \,\mathbf{curl}\,\boldsymbol{\rho}^{k-1} \cdot \mathbf{curl}\,\boldsymbol{\delta}^{k-1}. \end{aligned} \tag{3.56}$$

Hence, using (3.55) and (3.56) in (3.54), the Cauchy-Schwartz inequality leads to

$$\begin{aligned} &\Delta t \|\bar{\partial}\boldsymbol{\delta}^k\|_\sigma^2 + a(t_k; \boldsymbol{\delta}^k, \boldsymbol{\delta}^k) - a(t_{k-1}; \boldsymbol{\delta}^{k-1}, \boldsymbol{\delta}^{k-1}) \\ &\qquad \leq C_4 \Delta t \left[\|\bar{\partial}\boldsymbol{\rho}^k\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2 + \|\boldsymbol{\tau}^k\|_\sigma^2 + \|\,\mathbf{curl}\,\boldsymbol{\rho}^{k-1}\|_{0,\Omega}^2 \right] \\ &\qquad\quad - \left[a(t_k; \boldsymbol{\rho}^k, \boldsymbol{\delta}^k) - a(t_{k-1}; \boldsymbol{\rho}^{k-1}, \boldsymbol{\delta}^{k-1}) \right]. \end{aligned}$$

Summing over k and using the Cauchy-Schwartz inequality, we have

$$\Delta t \sum_{k=1}^{n} \|\bar{\partial}\boldsymbol{\delta}^k\|_\sigma^2 + \frac{1}{2\mu_1}\|\,\mathbf{curl}\,\boldsymbol{\delta}^n\|_{0,\Omega}^2$$
$$\leq C_5 \Delta t \left[\sum_{k=1}^{n} \|\bar{\partial}\boldsymbol{\rho}^k\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2 + \sum_{k=1}^{n} \|\boldsymbol{\tau}^k\|_\sigma^2 + \sum_{k=1}^{n} \|\,\mathbf{curl}\,\boldsymbol{\rho}^k\|_{0,\Omega}^2\right].$$

Finally, the result follows by combining the last inequality with (3.53). □

Theorem 3.6.1 *Assume that* $\boldsymbol{u} \in \mathrm{H}^2(0,T;\mathbf{H}(\mathbf{curl};\Omega))$ *and let* $\boldsymbol{e}^n := \boldsymbol{u}(t_n) - \boldsymbol{u}_h^n$*. Then, there exists a constant* $C > 0$*, independent of* h *and* Δt*, such that*

$$\max_{1\leq n\leq N} \|\boldsymbol{e}^n\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2 + \Delta t \sum_{k=1}^{N} \|\boldsymbol{e}^k\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2 + \Delta t \sum_{k=1}^{N} \|\bar{\partial}\boldsymbol{e}^k\|_\sigma^2$$
$$\leq C \left\{ \max_{1\leq n\leq N} \inf_{\boldsymbol{v}\in X_h(\Omega)} \|\boldsymbol{u}(t_n) - \boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2 + \Delta t \sum_{n=1}^{N} \inf_{\boldsymbol{v}\in X_h(\Omega)} \|\boldsymbol{u}(t_n) - \boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2 \right.$$
$$\left. + \int_0^T \left(\inf_{\boldsymbol{v}\in X_h(\Omega)} \|\partial_t \boldsymbol{u}(t) - \boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2\right) dt + \Delta t^2 \int_0^T \|\partial_{tt}\boldsymbol{u}(t)\|_\sigma^2 \, dt \right\}.$$

Proof. A Taylor expansion shows that

$$\sum_{k=1}^{n} \|\boldsymbol{\tau}^k\|_\sigma^2 = \sum_{k=1}^{n} \left\| \frac{1}{\Delta t} \int_{t_{k-1}}^{t_k} (t_{k-1} - t)\partial_{tt}\boldsymbol{u}(t)\, dt \right\|_\sigma^2 \leq \Delta t \int_0^T \|\partial_{tt}\boldsymbol{u}(t)\|_\sigma^2 \, dt. \quad (3.57)$$

Moreover,

$$\sum_{k=1}^{n} \|\bar{\partial}\boldsymbol{\rho}^k\|_\sigma^2 \leq \frac{1}{\Delta t} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \|\partial_t \boldsymbol{\rho}_h(t)\|_\sigma^2 \, dt \leq \frac{1}{\Delta t} \int_0^T \|\partial_t \boldsymbol{\rho}_h(t)\|_\sigma^2 \, dt. \quad (3.58)$$

Combining (3.50), (3.57), and (3.58) and recalling that $\|\cdot\|_{V_0(\Omega)}$ is equivalent to $\|\cdot\|_{\mathbf{H}(\mathbf{curl};\Omega)}$ in $V_{0,h}(\Omega)$, we obtain

$$\max_{1\leq n\leq N} \|\boldsymbol{\delta}^n\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2 + \Delta t \sum_{k=1}^{N} \|\boldsymbol{\delta}^k\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2 + \Delta t \sum_{k=1}^{N} \|\bar{\partial}\boldsymbol{\delta}^k\|_\sigma^2$$
$$\leq C_0 \left\{ \int_0^T \|\partial_t \boldsymbol{\rho}_h(t)\|_\sigma^2 \, dt + \Delta t \sum_{k=1}^{N} \|\,\mathbf{curl}\,\boldsymbol{\rho}_h(t_k)\|_{0,\Omega}^2 + (\Delta t)^2 \int_0^T \|\partial_{tt}\boldsymbol{u}(s)\|_\sigma^2 \, ds \right\}.$$

The result follows from the fact that $\boldsymbol{e}^n = \boldsymbol{\delta}^n + \boldsymbol{\rho}^n$ and the triangle inequality. □

Finally we deduce from (3.45), (3.46), and (3.48) the following asymptotic error estimate.

Corollary 3.6.1 *Under the assumptions of Corollary 3.5.1 and Theorem 3.6.1, there exists a constant C, independent of h and Δt, such that*

$$\max_{1\leq n\leq N}\|\boldsymbol{e}^n\|^2_{\mathbf{H}(\mathbf{curl};\Omega)}+\Delta t\sum_{k=1}^{N}\|\boldsymbol{e}^k\|^2_{\mathbf{H}(\mathbf{curl};\Omega)}+\Delta t\sum_{k=1}^{N}\|\bar{\partial}\boldsymbol{e}^k\|^2_{\sigma}$$
$$\leq Ch^{2l}\left\{\max_{1\leq n\leq N}\|\boldsymbol{u}(t_n)\|^2_{\boldsymbol{\mathcal{X}}}+\int_0^T\|\partial_t\boldsymbol{u}(t)\|^2_{\boldsymbol{\mathcal{X}}}\,dt\right\}$$
$$+C(\Delta t)^2\int_0^T\|\partial_{tt}\boldsymbol{u}(t)\|^2_{\sigma}\,dt,$$

with $l:=\min\{m,r\}$.

Remark 3.6.1 *At each time step $t=t_k$, we can approximate the eddy currents $\sigma\boldsymbol{E}(\boldsymbol{x},t_k)$ by $\sigma\boldsymbol{E}_h^k$, where $\boldsymbol{E}_h^k:=\bar{\partial}\boldsymbol{u}_h^k$. In fact, Corollary 3.6.1 yields the following convergence estimate in a discrete* $\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega_{\mathrm{c}}))$*-norm*

$$\Delta t\sum_{k=1}^{N}\|\sigma\boldsymbol{E}(t_k)-\sigma\boldsymbol{E}_h^k\|^2_{0,\Omega_{\mathrm{c}}}\leq C\left[h^{2l}+(\Delta t)^2\right].$$

3.7 Conclusions.

We have introduced an $\boldsymbol{E}$-based formulation for the time-dependent eddy current problem in a bounded domain. The variables of the formulation are a time-primitive of the electric field and a Lagrange multiplier used to impose the divergence-free constraint in the dielectric domain. We have shown that this formulation is well posed and that the Lagrange multiplier vanishes identically.

Then, we have proposed a finite element space discretization based on Nédélec edge elements for the main variable and standard nodal finite elements for the Lagrange multiplier. We have proved the well posedness of the resulting semi-discrete scheme as well as optimal order error estimates. The discrete Lagrange multiplier has been proved to vanish, as well. Finally we have analyzed an implicit time discretization scheme. Under appropriate smoothness assumptions, we have proved that the fully discrete problem also converges with optimal order. This approach provides suitable approximations of the quantities of typical interest: the eddy currents in the electric domain and the magnetic induction.

Chapter 4

Numerical treatment of a nonlinear magnetic field time-dependent eddy current problem

4.1 Introduction

The field problems involving ferromagnetic conducting materials are complicated by the nonlinear relationship between flux density and the magnetic field intensity. This relationship is given by a physical parameter called *magnetic reluctivity* (the inverse of the permeability). If hysteresis effects and anisotropies are neglected, the reluctivity is a scalar function which typically has a nonlinear dependence on the absolute value of the magnetic induction; see, for instance [44].

In the previous chapter, it has been shown that the time-dependent eddy current problem admits a saddle point structure, in which the reluctivity appears as a diffusion coefficient in the resulting degenerate parabolic problem (see (3.16)). Consequently, this formulation allows us to consider the above-mentioned nonlinear case.

By using the physical properties of the reluctivity (cf. [65]) and the nonlinear monotone operators theory, we prove that the resulting mixed problem is well-posed. As in Chapter 3, we use Nédélec edge elements for the main variable combined with standard nodal finite elements for the Lagrange multiplier to define a semi-discrete Galerkin scheme. Furthermore, we introduce the corresponding backward-Euler fully-discrete formulation and prove error estimates.

Throughout this chapter we will use the definitions and notations from Section 3.2.

4.2 Variational formulation

Our purpose is to determine the eddy currents induced by a given time-dependent compactly-supported current density $\boldsymbol{J}$ in a three-dimensional conducting domain represented by the open and bounded set Ω_{c}. We assume that Ω_{c} is a Lipschitz domain and denote by $\boldsymbol{n}$ the unit outward normal on $\Sigma := \partial\Omega_{\mathrm{c}}$. We denote by Σ_i, $i = 1, \cdots, I$, the connected components of Σ.

The electric and magnetic fields $\boldsymbol{E}(\boldsymbol{x}, t)$ and $\boldsymbol{H}(\boldsymbol{x}, t)$ are solutions of a submodel of Maxwell's equations obtained by neglecting the displacement currents (see [13]):

$$\partial_t (\mu \boldsymbol{H}) + \mathbf{curl}\, \boldsymbol{E} = \mathbf{0} \qquad \text{in } \mathbb{R}^3 \times (0, T), \tag{4.1}$$

$$\mathbf{curl}\, \boldsymbol{H} = \boldsymbol{J} + \sigma \boldsymbol{E} \qquad \text{in } \mathbb{R}^3 \times [0, T), \tag{4.2}$$

$$\operatorname{div}(\varepsilon \boldsymbol{E}) = 0 \qquad \text{in } \left(\mathbb{R}^3 \setminus \Omega_{\mathrm{c}}\right) \times [0, T), \tag{4.3}$$

$$\int_{\Sigma_i} \varepsilon \boldsymbol{E} \cdot \boldsymbol{n} = 0 \qquad \text{in } [0, T), \quad i = 1, \cdots, I, \tag{4.4}$$

$$\boldsymbol{H}(\boldsymbol{x}, 0) = \mathbf{0} \qquad \text{in } \mathbb{R}^3, \tag{4.5}$$

$$\boldsymbol{H}(\boldsymbol{x}, t) = O\left(\frac{1}{|\boldsymbol{x}|}\right) \quad \text{and} \quad \boldsymbol{E}(\boldsymbol{x}, t) = O\left(\frac{1}{|\boldsymbol{x}|}\right) \qquad \text{as } |\boldsymbol{x}| \to \infty, \tag{4.6}$$

where the asymptotic behavior (4.6) holds uniformly in $(0, T)$. Since we only consider here isotropic materials, the electric permittivity ε, the electric conductivity σ and the magnetic permeability μ are piecewise smooth real valued functions.

In ferromagnetic materials (e.g. iron, steel) the magnetic permeability μ does not only depend on the space variable. Actually, disregarding the effects of hysteresis, μ is a function of the magnetic field ($\mu = \mu(|\boldsymbol{H}|)$). Then, by introducing the magnetic reluctivity

$$\nu = \nu(|\boldsymbol{B}|) = \frac{1}{\mu(|\boldsymbol{H}|)} \qquad (\boldsymbol{B} := \mu \boldsymbol{H}),$$

we have the following nonlinear relation between the magnetic induction $\boldsymbol{B}$ and the magnetic field $\boldsymbol{H}$ (c.f. [44])

$$\boldsymbol{H} = \nu(|\boldsymbol{B}|)\boldsymbol{B}. \tag{4.7}$$

By proceeding as in Section 3.3, we will formulate our problem in terms of the time primitive of the electric field

$$\boldsymbol{u}(\boldsymbol{x}, t) := \int_0^t \boldsymbol{E}(\boldsymbol{x}, s)\, ds.$$

To this end, we integrate (4.1) with respect to t and use (4.5), to obtain

$$\boldsymbol{B} = -\,\mathbf{curl}\,\boldsymbol{u}. \tag{4.8}$$

Hence, (4.7) implies

$$\boldsymbol{H} = -\nu(|\mathbf{curl}\,\boldsymbol{u}|)\,\mathbf{curl}\,\boldsymbol{u}.$$

Then, from (4.2) we have

$$\sigma\partial_t \boldsymbol{u} + \mathbf{curl}\,(\nu(|\mathbf{curl}\,\boldsymbol{u}|)\,\mathbf{curl}\,\boldsymbol{u}) = -\boldsymbol{J}.$$

Notice that as a consequence of the decay conditions (4.6), we may assume that the electromagnetic field is weak far away from Ω_{c}. Motivated by this fact, and aiming to obtaining a suitable simplification of our model problem, we introduce a closed surface Γ located sufficiently far from $\overline{\Omega}_{\mathrm{c}}$ and assume that $\boldsymbol{u}$ has a vanishing tangential trace on this surface. Hence, we will formulate our problem in the bounded domain Ω with boundary Γ. We assume that Ω is simply connected, with a connected boundary, and that it contains Ω_{c} and the support of $\boldsymbol{J}$. We define $\Omega_{\mathrm{d}} := \Omega \setminus \overline{\Omega}_{\mathrm{c}}$.

The last considerations lead us to the following formulation of the eddy current problem:
Find $\boldsymbol{u} : \Omega \times [0, T) \to \mathbb{R}^3$ such that:

$$\begin{aligned} \sigma\partial_t \boldsymbol{u} + \mathbf{curl}\,(\nu(|\mathbf{curl}\,\boldsymbol{u}|)\,\mathbf{curl}\,\boldsymbol{u}) &= -\boldsymbol{J} && \text{in } \Omega \times (0, T), \\ \operatorname{div}(\varepsilon \boldsymbol{u}) &= 0 && \text{in } \Omega_{\mathrm{d}} \times [0, T), \\ \langle \gamma_n(\varepsilon \boldsymbol{u}), 1\rangle_{\Sigma_i} &= 0 && \text{in } [0, T), \quad i = 1, \dots, I, \\ \gamma_\tau \boldsymbol{u} &= \mathbf{0} && \text{on } \Gamma \times [0, T), \\ \boldsymbol{u}(\cdot, 0) &= \mathbf{0} && \text{in } \Omega. \end{aligned} \tag{4.9}$$

We introduce the space

$$M(\Omega_{\mathrm{d}}) := \left\{ \vartheta \in \mathrm{H}^1(\Omega_{\mathrm{d}}) : \ \gamma\vartheta|_{\Gamma} = 0 \text{ and } \gamma\vartheta|_{\Sigma_i} = C_i, \ i = 1, \dots, I \right\},$$

where C_i, $i = 1, \dots, I$, are arbitrary constants. The Poincaré inequality shows that $|\cdot|_{1,\Omega_{\mathrm{d}}}$ is a norm on $M(\Omega_{\mathrm{d}})$ equivalent to the usual $\mathrm{H}^1(\Omega_{\mathrm{d}})$-norm. Next, let

$$V_0(\Omega) := \{\boldsymbol{v} \in \mathbf{H}_0(\mathbf{curl}, \Omega) : \ b(\boldsymbol{v}, \vartheta) = 0 \quad \forall \vartheta \in M(\Omega_{\mathrm{d}})\}, \tag{4.10}$$

where

$$b(\boldsymbol{u}, \vartheta) := (\varepsilon \boldsymbol{u}, \mathbf{grad}\, \vartheta)_{0,\Omega_{\mathrm{d}}}.$$

By testing the first equation of (3.12) with a function $\boldsymbol{v} \in V_0(\Omega)$ and using (3.1), we obtain the following variational formulation:
Find $\boldsymbol{u} \in \mathcal{W}_0$ such that

$$\begin{aligned} \frac{d}{dt}(\sigma \boldsymbol{u}(t), \boldsymbol{v})_{0,\Omega_{\mathrm{c}}} + (\nu(|\mathbf{curl}\, \boldsymbol{u}|)\, \mathbf{curl}\, \boldsymbol{u}(t), \mathbf{curl}\, \boldsymbol{v})_{0,\Omega} = (\boldsymbol{J}(t), \boldsymbol{v})_{0,\Omega} \qquad & \forall \boldsymbol{v} \in V_0(\Omega), \\ \boldsymbol{u}|_{\Omega_{\mathrm{c}}}(0) = \mathbf{0}, & \end{aligned} \tag{4.11}$$

where

$$\mathcal{W}_0 := \{\boldsymbol{v} \in \mathrm{L}^2(0, T; V_0(\Omega)) : \ \boldsymbol{v}|_{\Omega_{\mathrm{c}}} \in W^1(0, T; \mathbf{H}(\mathbf{curl}, \Omega_{\mathrm{c}}))\},$$

with

$$\begin{aligned} & W^1(0, T; \mathbf{H}(\mathbf{curl}, \Omega_{\mathrm{c}})) \\ & \qquad := \left\{\boldsymbol{v} \in \mathrm{L}^2(0, T; \mathbf{H}(\mathbf{curl}, \Omega_{\mathrm{c}})) : \ \partial_t \boldsymbol{v} \in \mathrm{L}^2(0, T; \mathbf{H}(\mathbf{curl}, \Omega_{\mathrm{c}})')\right\}. \end{aligned}$$

Here, $\mathbf{H}(\mathbf{curl}, \Omega_{\mathrm{c}})'$ is the dual space of $\mathbf{H}(\mathbf{curl}, \Omega_{\mathrm{c}})$ with respect to the pivot space

$$\mathrm{L}^2(\Omega_{\mathrm{c}}, \sigma)^3 := \left\{\boldsymbol{v} : \Omega_{\mathrm{c}} \to \mathbb{R}^3 \text{ Lebesgue-measurable} : \ \int_{\Omega_{\mathrm{c}}} \sigma |\boldsymbol{v}|^2 < \infty\right\}.$$

Notice that the initial condition of (4.11) makes sense thanks to the continuous embedding (c.f. [73, Proposition 23.23]):

$$W^1(0, T; \mathbf{H}(\mathbf{curl}, \Omega_{\mathrm{c}})) \hookrightarrow \mathcal{C}^0(0, T; \mathrm{L}^2(\Omega_{\mathrm{c}}, \sigma)^3).$$

In order to avoid the task of constructing a conforming finite element discretization of (4.11), we take advantage of Lemma 3.3.1 and propose a mixed formulation of the problem. To this end, we relax as follows the divergence-free restriction through a Lagrange multiplier:

Find $\boldsymbol{u} \in \mathcal{W}$ and $\lambda \in \mathrm{L}^2(0,T;M(\Omega_\mathrm{d}))$ such that:

$$\begin{aligned} \frac{d}{dt}\left((\boldsymbol{u}(t),\boldsymbol{v})_\sigma + b(\boldsymbol{v},\lambda(t))\right) + \langle A\boldsymbol{u}(t),\boldsymbol{v}\rangle &= (-\boldsymbol{J}(t),\boldsymbol{v})_{0,\Omega} \quad \forall \boldsymbol{v} \in \mathbf{H}_0(\mathbf{curl},\Omega), \\ b(\boldsymbol{u}(t),\vartheta) &= 0 \quad \forall \vartheta \in M(\Omega_\mathrm{d}), \\ \boldsymbol{u}|_{\Omega_\mathrm{c}}(0) &= \boldsymbol{0}, \end{aligned} \tag{4.12}$$

where,

$$\mathcal{W} := \left\{ \boldsymbol{v} \in \mathrm{L}^2(0,T;\mathbf{H}_0(\mathbf{curl},\Omega)) :\ \boldsymbol{v}|_{\Omega_\mathrm{c}} \in W^1(0,T;\mathbf{H}(\mathbf{curl},\Omega_\mathrm{c})) \right\},$$
$$(\boldsymbol{u},\boldsymbol{v})_\sigma := (\sigma\boldsymbol{u},\boldsymbol{v})_{0,\Omega_\mathrm{c}}, \qquad \langle A\boldsymbol{u},\boldsymbol{v}\rangle := \int_\Omega \nu(|\mathbf{curl}\,\boldsymbol{u}|)\,\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v}.$$

4.3 Existence and uniqueness of weak solutions

From now on we assume that the functions ε and σ satisfy the following (physical) properties:

$$\varepsilon_1 \geq \varepsilon(\boldsymbol{x}) \geq \varepsilon_0 > 0 \quad \text{a.e. in } \Omega_\mathrm{c} \qquad \text{and} \qquad \varepsilon(\boldsymbol{x}) = \varepsilon_0 \quad \text{a.e. in } \Omega_\mathrm{d}, \tag{4.13}$$
$$\sigma_1 \geq \sigma(\boldsymbol{x}) \geq \sigma_0 > 0 \quad \text{a.e. in } \Omega_\mathrm{c} \qquad \text{and} \qquad \sigma(\boldsymbol{x}) = 0 \quad \text{a.e. in } \Omega_\mathrm{d}. \tag{4.14}$$

Notice that, as a consequence of (4.2) and (4.14), $\boldsymbol{J}$ is divergence-free in $\mathbb{R}^3 \setminus \Omega_\mathrm{c}$ and $\int_{\Sigma_i} \boldsymbol{J}\cdot\boldsymbol{n} = 0$, $i = 1,\cdots,I$, for all $t \in [0,T)$. Then

$$(\boldsymbol{J},\mathbf{grad}\,\vartheta)_{0,\Omega_\mathrm{d}} = \langle \gamma_n\boldsymbol{J},\vartheta\rangle_{\partial\Omega_\mathrm{d}} = \sum_{i=1}^{I} \vartheta|_{\Sigma_i}\langle\gamma_n\boldsymbol{J},1\rangle_{\Sigma_i} = 0 \qquad \forall\vartheta \in M(\Omega_\mathrm{d}). \tag{4.15}$$

Furthermore, we suppose that there exists a constant $\nu_1 > 0$ (the reluctivity of the dielectric medium) such that

$$\nu(|\mathbf{curl}\,\boldsymbol{u}(\boldsymbol{x},t)|) = \begin{cases} \nu_1, & \boldsymbol{x} \in \Omega_\mathrm{d},\ t \in (0,T), \\ \nu_\mathrm{c}(|\mathbf{curl}\,\boldsymbol{u}(\boldsymbol{x},t)|), & \boldsymbol{x} \in \Omega_\mathrm{c},\ t \in (0,T), \end{cases} \tag{4.16}$$

where $\nu_\mathrm{c} : \mathbb{R}_0^+ \to \mathbb{R}^+$ is a continuous function satisfying the following properties (c.f. [65]): There exists constants $\nu_{\min}$ and ν_0 (with ν_0 the reluctivity in the vacuum) such that

$$\begin{aligned} 0 < \nu_{\min} \leq \nu_\mathrm{c}(s) \leq \nu_0 \qquad &\forall s \in \mathbb{R}_0^+, \\ \lim_{s\to\infty} \nu_\mathrm{c}(s) = \nu_0, & \end{aligned} \tag{4.17}$$

and

$$s \mapsto \nu_{\mathrm{c}}(s)s \text{ is strictly increasing and Lipschitz continuous.} \tag{4.18}$$

Let us remark that (4.16) implies

$$\langle A\boldsymbol{u}, \boldsymbol{v}\rangle = \langle B\boldsymbol{u}, \boldsymbol{v}\rangle + a(\boldsymbol{u}, \boldsymbol{v}), \tag{4.19}$$

where

$$\langle B\boldsymbol{u}, \boldsymbol{v}\rangle := \int_{\Omega_{\mathrm{c}}} \nu_{\mathrm{c}}(|\,\mathbf{curl}\,\boldsymbol{u}|)\,\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v} \tag{4.20}$$

and

$$a(\boldsymbol{u}, \boldsymbol{v}) := \nu_1 \int_{\Omega_{\mathrm{d}}} \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v}. \tag{4.21}$$

We are now in a position to prove the main result of this section.

Theorem 4.3.1 *Problem* (4.12) *has a unique solution* $(\boldsymbol{u}, \lambda)$ *with* $\lambda = 0$. *Furthermore, there exists* $C > 0$ *such that*

$$\max_{t\in[0,T]} \|\boldsymbol{u}(t)\|^2_{0,\Omega_{\mathrm{c}}} + \int_0^T \|\boldsymbol{u}(t)\|^2_{\mathbf{H}(\mathbf{curl};\Omega)}\,dt \le C\int_0^T \|\boldsymbol{f}(t)\|^2_{0,\Omega}\,dt. \tag{4.22}$$

Proof. We first notice that the second equation of (4.12) means that $\boldsymbol{u} \in \mathcal{W}_0$. Let $V_0(\Omega_{\mathrm{d}})$ the subspace defined by (3.17). The decomposition (3.23) implies that the direct sum

$$\mathcal{W}_0 = \mathrm{L}^2(0,T;\widetilde{V_0(\Omega_{\mathrm{d}})}) \oplus \mathcal{E}(W^1(0,T;\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}})))$$

is orthogonal with respect to the inner product $\int_0^T (\cdot,\cdot)_{V_0(\Omega)}\,dt$. Hence $\boldsymbol{u} = \boldsymbol{u}_{\mathrm{d}} + \mathcal{E}\boldsymbol{u}_{\mathrm{c}}$, with $\boldsymbol{u}_{\mathrm{d}} \in \mathrm{L}^2(0,T;\widetilde{V_0(\Omega_{\mathrm{d}})})$ and $\mathcal{E}\boldsymbol{u}_{\mathrm{c}} \in \mathcal{E}(W^1(0,T;\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}})))$. Testing the first equation of (4.12) with $\boldsymbol{v} \in \widetilde{V_0(\Omega_{\mathrm{d}})}$, we find that the first component satisfies

$$\nu_1\,(\mathbf{curl}\,\boldsymbol{u}_{\mathrm{d}}(t), \mathbf{curl}\,\boldsymbol{v})_{0,\Omega_{\mathrm{d}}} = (-\boldsymbol{J}(t), \boldsymbol{v})_{0,\Omega_{\mathrm{d}}} \qquad \forall \boldsymbol{v} \in V_0(\Omega_{\mathrm{d}}).$$

The existence and uniqueness of solution of this problem is a consequence of the Lax-Milgram lemma and Lemma 3.4.1. Moreover, there exists $C_1 > 0$ such that

$$\int_0^T \|\boldsymbol{u}_{\mathrm{d}}\|^2_{\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{d}})}\,dt \le C_1 \int_0^T \|\boldsymbol{J}(t)\|^2_{0,\Omega}\,dt. \tag{4.23}$$

The other component is determined by solving

$$\begin{aligned} &\frac{d}{dt}(\boldsymbol{u}_{\mathrm{c}}(t), \boldsymbol{v})_\sigma + \langle A\mathcal{E}\boldsymbol{u}_{\mathrm{c}}(t), \mathcal{E}\boldsymbol{v}\rangle = (\boldsymbol{J}(t), \mathcal{E}\boldsymbol{v})_{0,\Omega} \qquad \forall \boldsymbol{v} \in \mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}}), \\ &\boldsymbol{u}_{\mathrm{c}}(0) = \mathbf{0}. \end{aligned} \tag{4.24}$$

In order to prove that the last problem has a unique solution, we notice that the first equation of (4.24) is equivalent to

$$\frac{d}{dt}(\boldsymbol{w}(t),\boldsymbol{v})_\sigma + \langle C(t)\boldsymbol{w}(t),\boldsymbol{v}\rangle = -e^t\left(\boldsymbol{J}(t),\mathcal{E}\boldsymbol{v}\right)_{0,\Omega} \qquad \forall \boldsymbol{v}\in \mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}}),$$

where $\boldsymbol{w}(t) = e^t\boldsymbol{u}_{\mathrm{c}}(t)$ and $C(t) : \mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}}) \to \mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}})'$ is the nonlinear operator defined by

$$\begin{aligned}\langle C(t)\boldsymbol{w},\boldsymbol{v}\rangle :=&(\boldsymbol{w},\boldsymbol{v})_\sigma + \left(\nu_{\mathrm{c}}(e^t|\mathbf{curl}\,\boldsymbol{w}|)\,\mathbf{curl}\,\boldsymbol{w},\mathbf{curl}\,\boldsymbol{v}\right)_{0,\Omega_{\mathrm{c}}}\\ &+\nu_1\left(\mathbf{curl}\,\mathcal{E}\boldsymbol{w},\mathbf{curl}\,\mathcal{E}\boldsymbol{v}\right)_{0,\Omega_{\mathrm{d}}}.\end{aligned} \tag{4.25}$$

Consequently, it is sufficient to show the following properties for any $t\in(0,T)$ (c.f. [74, Theorem 30A]):

C1. $C(t)$ is *Hemicontinuous*, i.e. the function

$$s\mapsto \langle C(t)(\boldsymbol{u}+s\boldsymbol{v}),\boldsymbol{w}\rangle \tag{4.26}$$

is continuous in $[0,1]$ for any $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}})$.

C2. $C(t)$ is *Monotone*, i.e. there holds

$$\langle C(t)\boldsymbol{v}-C(t)\boldsymbol{w},\boldsymbol{v}-\boldsymbol{w}\rangle \geq 0 \qquad \forall \boldsymbol{v},\boldsymbol{w}\in\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}}). \tag{4.27}$$

C3. $C(t)$ is *Coercive*, i.e. there exists $\kappa>0$ such that

$$\langle C(t)\boldsymbol{v},\boldsymbol{v}\rangle \geq \kappa\|\boldsymbol{v}\|^2_{\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}})} \qquad \forall \boldsymbol{v}\in\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}}).$$

C4. $C(t)$ is *Bounded*, i.e. there exists $\tilde{\kappa}>0$ such that

$$\|C(t)\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}})'} \leq \tilde{\kappa}\|\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}})} \qquad \forall \boldsymbol{v}\in\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}}). \tag{4.28}$$

By using (4.17) and (4.18), it is straightforward to prove the properties **C1-C4** (see remark below). Then, there exists a unique $\boldsymbol{u}_{\mathrm{c}}$ solution of Problem (4.24) satisfying

$$\max_{t\in[0,T]}\|\boldsymbol{u}_{\mathrm{c}}(t)\|^2_{0,\Omega_{\mathrm{c}}} + \int_0^T \|\boldsymbol{u}_{\mathrm{c}}(t)\|^2_{\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{d}})}\,dt \leq C_2\int_0^T \|\boldsymbol{J}(t)\|^2_{0,\Omega}\,dt,$$

for some constant $C_2>0$. Combining the last inequality with (4.23) we obtain (4.22).

To show the existence of the Lagrange multiplier λ, we consider the functional $\mathcal{G} \in C^0([0,T];\mathbf{H}_0(\mathbf{curl},\Omega)')$ defined by

$$\langle \mathcal{G}(t), \boldsymbol{v}\rangle := -(\boldsymbol{u}(t),\boldsymbol{v})_\sigma - \int_0^t \langle A\boldsymbol{u}(s),\boldsymbol{v}\rangle\, ds - \int_0^t (\boldsymbol{J}(s),\boldsymbol{v})_{0,\Omega}\, ds$$

for all $\boldsymbol{v} \in \mathbf{H}_0(\mathbf{curl},\Omega)$ and proceed as in Theorem 3.4.1. Finally, by proceeding as in Lemma 3.4.4 we prove that λ vanishes identically . □

Remark 4.3.1 *For the sake of completeness, we present here the proof of properties* **C1-C4** *of operator* (4.25)*. Let $t \in]0,T[$, $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \mathbf{H}(\mathbf{curl},\Omega_c)$.*

Proof of **C1***. Let $f :]0,T[\to \mathbb{R}$ be the function given by* (4.26)*. For any $s \in]0,T[$ we have*

$$\begin{aligned} f(s) =&(\boldsymbol{u}+s\boldsymbol{v},\boldsymbol{w})_\sigma + \int_{\Omega_c} \nu_c(e^t|\mathbf{curl}(\boldsymbol{u}+s\boldsymbol{v})|)\,\mathbf{curl}(\boldsymbol{u}+s\boldsymbol{v})\cdot\mathbf{curl}\,\boldsymbol{w} \\ &+ \nu_1(\mathbf{curl}\,\mathcal{E}(\boldsymbol{u}+s\boldsymbol{v}),\mathbf{curl}\,\mathcal{E}\boldsymbol{w})_{0,\Omega_d}. \end{aligned}$$

Then, since ν_c continuous, we deduce that f is continuous.

Proof of **C2***. Let $D : \mathbf{H}(\mathbf{curl},\Omega_c) \to \mathbf{H}(\mathbf{curl},\Omega_c)'$ and $g : [0,1] \to \mathbb{R}$ be the functions defined by*

$$\langle D\boldsymbol{v},\boldsymbol{w}\rangle := \int_{\Omega_c} \nu_c(e^t|\mathbf{curl}\,\boldsymbol{v}|)\,\mathbf{curl}\,\boldsymbol{v}\cdot\mathbf{curl}\,\boldsymbol{w} \tag{4.29}$$

and

$$g(s) := \langle D(s\boldsymbol{v}+(1-s)\boldsymbol{w}),\boldsymbol{v}-\boldsymbol{w}\rangle, \tag{4.30}$$

respectively. To prove (4.27)*, it is sufficient to show that*

$$g(1)-g(0) = \int_0^1 g'(s)ds = \langle D\boldsymbol{v}-D\boldsymbol{w},\boldsymbol{v}-\boldsymbol{w}\rangle \geq 0. \tag{4.31}$$

Let $\boldsymbol{\alpha} := e^t\,\mathbf{curl}\,\boldsymbol{u}$, $\boldsymbol{\beta} := e^t\,\mathbf{curl}\,\boldsymbol{w}$ and

$$G(s) := \nu_c(|s\boldsymbol{\alpha}+(1-s)\boldsymbol{\beta}|)(s\boldsymbol{\alpha}+(1-s)\boldsymbol{\beta})\cdot(\boldsymbol{\alpha}-\boldsymbol{\beta}). \tag{4.32}$$

Then,

$$g(s) = e^{-2t}\int_{\Omega_c} G(s). \tag{4.33}$$

By differentiating (4.32) *with respect to s, we have*

$$\begin{aligned} G'(s) =& \nu_c'(|s\boldsymbol{\alpha}+(1-s)\boldsymbol{\beta}|)\left[\frac{|(s\boldsymbol{\alpha}+(1-s)\boldsymbol{\beta})\cdot(\boldsymbol{\alpha}-\boldsymbol{\beta})|^2}{|s\boldsymbol{\alpha}+(1-s)\boldsymbol{\beta}|}\right] \\ &+ \nu_c(|s\boldsymbol{\alpha}+(1-s)\boldsymbol{\beta}|)|\boldsymbol{\alpha}-\boldsymbol{\beta}|^2 \\ =& \left\{[\nu_c'(\eta)\eta+\nu_c(\eta)]\cos^2\theta + \nu_c(\eta)\sin^2\theta\right\}|\boldsymbol{\alpha}-\boldsymbol{\beta}|^2, \end{aligned}$$

where

$$\cos\theta := \frac{(s\boldsymbol{\alpha} + (1-s)\boldsymbol{\beta})\cdot(\boldsymbol{\alpha}-\boldsymbol{\beta})}{|s\boldsymbol{\alpha} + (1-s)\boldsymbol{\beta}||\boldsymbol{\alpha}-\boldsymbol{\beta}|}, \qquad \eta := |s\boldsymbol{\alpha} + (1-s)\boldsymbol{\beta}|.$$

Then, recalling that $\eta \mapsto \nu_{\rm c}(\eta)\eta$ *is strictly increasing, we obtain that* $G'(s) \geq 0$ *for any* $s \in (0,1)$. *Hence, by* (4.33) *we conclude* (4.31).

Proof of **C3**. *It suffices to notice that*

$$\begin{aligned}\langle C(t)\boldsymbol{v},\boldsymbol{v}\rangle =& \|\boldsymbol{v}\|^2_{\sigma,\Omega_{\rm c}} + \langle D\boldsymbol{v},\boldsymbol{v}\rangle + \nu_1\|\mathbf{curl}\,\mathcal{E}\boldsymbol{v}\|^2_{0,\Omega_{\rm d}}\\ \geq& \|\boldsymbol{v}\|^2_{\sigma,\Omega_{\rm c}} + \langle D\boldsymbol{v},\boldsymbol{v}\rangle\\ \geq& \sigma_0\|\boldsymbol{v}\|^2_{0,\Omega_{\rm c}} + \nu_{\min}\|\,\mathbf{curl}\,\boldsymbol{v}\|^2_{0,\Omega_{\rm c}}\\ \geq& \kappa\|\boldsymbol{v}\|^2_{\mathbf{H}(\mathbf{curl},\Omega_{\rm c})},\end{aligned}$$

where $\kappa := \min\{\sigma_0, \nu_{\min}\}$.

Proof of **C4**. *First observe that*

$$\begin{aligned}\langle C(t)\boldsymbol{v},\boldsymbol{w}\rangle \leq& \sigma_1\|\boldsymbol{v}\|_{0,\Omega_{\rm c}}\|\boldsymbol{w}\|_{0,\Omega_{\rm c}}\\ &+ \nu_0\|\,\mathbf{curl}\,\boldsymbol{v}\|_{0,\Omega_{\rm c}}\|\,\mathbf{curl}\,\boldsymbol{w}\|_{0,\Omega_{\rm c}} + \nu_1\|\,\mathbf{curl}\,\mathcal{E}\boldsymbol{v}\|_{0,\Omega_{\rm d}}\|\,\mathbf{curl}\,\mathcal{E}\boldsymbol{w}\|_{0,\Omega_{\rm d}}.\end{aligned}$$

Since $\mathcal{E}$ *is bounded, this inequality implies* (4.28).

4.4 Analysis of the semi-discrete scheme

In what follows we assume that Ω and $\Omega_{\rm c}$ are Lipschitz polyhedra. Let $\{\mathcal{T}_h\}_h$ be a regular family of tetrahedral meshes of Ω such that each element $K \in \mathcal{T}_h$ is contained either in $\overline{\Omega}_{\rm c}$ or in $\overline{\Omega}_{\rm d}$. As usual, h stands for the largest diameter of the tetrahedra K in $\mathcal{T}_h$. Furthermore, we suppose that the family of triangulations $\{\mathcal{T}_h(\Sigma)\}_h$ induced by $\{\mathcal{T}_h\}_h$ on Σ is quasi-uniform.

We define a semi-discrete version of (4.12) by means of Nédélec finite elements. The local representation of the mth-order element of this family on a tetrahedron K is given by (see [57, Section 5.5])

$$\mathcal{N}_m(K) := \mathbb{P}^3_{m-1} \oplus S_m,$$

where $\mathbb{P}_m$ is the set of polynomials of degree not greater than m and

$$S_m := \left\{p \in \widetilde{\mathbb{P}}^3_m : \ \boldsymbol{x}\cdot p(\boldsymbol{x}) = 0\right\},$$

with $\widetilde{\mathbb{P}}_m$ being the set of homogeneous polynomials of degree m. The corresponding global space $X_h(\Omega)$ is the space of functions that are locally in $\mathcal{N}_m(K)$ and have continuous tangential components across the faces of the triangulation $\mathcal{T}_h$:

$$X_h(\Omega) := \{\boldsymbol{v} \in \mathbf{H}_0(\mathbf{curl}, \Omega) : \ \boldsymbol{v}|_K \in \mathcal{N}_m(K) \ \forall K \in \mathcal{T}\}.$$

The degrees of freedom of $X_h(\Omega)$ are given by

$$\mathcal{M}_1(\boldsymbol{v}) := \left\{ \int_e \boldsymbol{v} \cdot \boldsymbol{t}_e \, q \text{ for all } q \in \mathbb{P}_{m-1} \text{ for the six edges } e \text{ of } K \right\}, \tag{4.34}$$

where $\boldsymbol{t}_e$ a unit tangent vector along e; when $m \geq 2$ one has to add

$$\mathcal{M}_2(\boldsymbol{v}) := \left\{ \int_f (\boldsymbol{v} \times \boldsymbol{n}) \cdot \boldsymbol{q} \text{ for all } \boldsymbol{q} \in \mathbb{P}^2_{m-2} \text{ for the four faces } f \text{ of } K \right\}; \tag{4.35}$$

and finally for $m \geq 3$ one has to take also

$$\mathcal{M}_3(\boldsymbol{v}) := \left\{ \int_K \boldsymbol{v} \cdot \boldsymbol{q} \text{ for all } \boldsymbol{q} \in \mathbb{P}^3_{m-3} \right\}. \tag{4.36}$$

Nédélec [61] has proven that these degrees of freedom are "curl-conforming" and determine a unique element of $\mathcal{N}_m(K)$.

We use standard mth-order Lagrange finite elements to approximate $M(\Omega_{\mathrm{d}})$:

$$M_h(\Omega_{\mathrm{d}}) := \left\{ \vartheta \in \mathrm{H}^1(\Omega_{\mathrm{d}}) : \ \vartheta|_K \in \mathbb{P}_m \ \forall K \in \mathcal{T}, \ \vartheta|_\Gamma = 0, \ \vartheta|_{\Sigma_i} = C_i, \ i = 1, \ldots, I \right\}.$$

We introduce the following semi-discretization of problem (4.12):
Find $\boldsymbol{u}_h(t) : [0, T] \to X_h(\Omega)$ and $\lambda_h(t) : [0, T] \to M_h(\Omega_{\mathrm{d}})$ such that

$$\begin{aligned} \frac{d}{dt} \left[(\boldsymbol{u}_h(t), \boldsymbol{v})_\sigma + b(\boldsymbol{v}, \lambda_h(t)) \right] + \langle A\boldsymbol{u}_h(t), \boldsymbol{v} \rangle &= - (\boldsymbol{J}(t), \boldsymbol{v})_{0,\Omega} \qquad \forall \boldsymbol{v} \in X_h(\Omega), \\ b(\boldsymbol{u}_h(t), \vartheta) &= 0 \qquad \forall \vartheta \in M_h(\Omega_{\mathrm{d}}), \\ \boldsymbol{u}_h|_{\Omega_{\mathrm{c}}}(0) &= \mathbf{0}. \end{aligned} \tag{4.37}$$

Theorem 4.4.1 *Problem* (4.37) *has a unique solution* $(\boldsymbol{u}_h, \lambda_h)$ *with an identically vanishing discrete Lagrange multiplier* λ_h.

Proof. The second equation of (4.37) means that $\boldsymbol{u}_h(t) \in \widetilde{V_{0,h}(\Omega)}$ for any $t \in (0, T)$. From (3.23), there holds that $\boldsymbol{u}_h = \boldsymbol{u}_{\mathrm{d},h} + \mathcal{E}_h(\boldsymbol{u}_{\mathrm{c},h})$ with $\boldsymbol{u}_{\mathrm{d},h}(t) \in V_{0,h}(\Omega_{\mathrm{d}})$ and $\boldsymbol{u}_{\mathrm{c},h}(t) \in X_h(\Omega_{\mathrm{c}})$. By proceeding as in Theorem 3.5.1, we notice that the existence and uniqueness of $\boldsymbol{u}_{\mathrm{d},h}$ is

a direct consequence of Lemma 3.5.1 and the Lax-Milgram lemma. The other component of the solution solves the following nonlinear finite system of differential equations
Find $\boldsymbol{u}_{\mathrm{c},h}(t) \in X_h(\Omega_\mathrm{c})$ such that:

$$\frac{d}{dt}\left(\boldsymbol{u}_{\mathrm{c},h}(t), \boldsymbol{v}\right)_\sigma + \langle A\mathcal{E}_h \boldsymbol{u}_{\mathrm{c},h}(t), \mathcal{E}_h \boldsymbol{v}\rangle = -\left(\boldsymbol{J}(t), \mathcal{E}_h \boldsymbol{v}\right)_{0,\Omega} \qquad \forall \boldsymbol{v} \in X_h(\Omega_\mathrm{c}),$$

$$\boldsymbol{u}_{\mathrm{c},h}(0) = \mathbf{0}.$$

To prove the existence and uniqueness of the Lagrange multiplier $\lambda_h = 0$, we define

$$\langle \mathcal{G}_h(t), \boldsymbol{v}\rangle := -\int_0^t \Big[(\boldsymbol{J}(s), \boldsymbol{v})_{0,\Omega} + \langle A\boldsymbol{u}_h(s), \boldsymbol{v}\rangle \Big]\, ds - (\boldsymbol{u}_h(t), \boldsymbol{v})_\sigma$$

and proceed as in Theorem 3.5.1. □

4.4.1 Error estimates

Our next goal is to prove error estimates for our semi-discrete scheme. Notice that as $\lambda = \lambda_h = 0$, we will only be concerned with error estimates for the main variable $\boldsymbol{u}$. To this end, we need to show some properties of the nonlinear operator A.

Lemma 4.4.1 *The operator A is hemicontinuous and bounded (see the proof of Theorem 3.4.1 for these definitions). Furthermore, there hold the following inequalities for any $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbf{H}(\mathbf{curl}; \Omega)$:*

$$\langle A\boldsymbol{v} - A\boldsymbol{w}, \boldsymbol{v} - \boldsymbol{w}\rangle \geq \alpha \| \mathbf{curl}(\boldsymbol{v} - \boldsymbol{w})\|_{0,\Omega}^2, \tag{4.38}$$

$$|\langle A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{w}\rangle| \leq \kappa \| \mathbf{curl}\, \boldsymbol{u} - \mathbf{curl}\, \boldsymbol{v}\|_{0,\Omega} \| \mathbf{curl}\, \boldsymbol{w}\|_{0,\Omega}, \tag{4.39}$$

where α and κ are positive constants.

Proof. The hemicontinuity, boundness and inequality (4.38) are straightforward (see Remark 4.3.1).

Let B the operator defined by (4.20). First we want to prove that there exists $\kappa_1 > 0$ such that

$$|\langle B\boldsymbol{u} - B\boldsymbol{v}, \boldsymbol{w}\rangle| \leq \kappa_1 \| \mathbf{curl}\, \boldsymbol{u} - \mathbf{curl}\, \boldsymbol{v}\|_{0,\Omega_\mathrm{c}} \| \mathbf{curl}\, \boldsymbol{w}\|_{0,\Omega_\mathrm{c}} \tag{4.40}$$

for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbf{H}(\mathbf{curl}, \Omega_\mathrm{c})$.

Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbf{H}(\mathbf{curl}, \Omega_{\rm c})$. It is easy to check that

$$\langle B\boldsymbol{u} - B\boldsymbol{v}, \boldsymbol{w}\rangle = \int_{\Omega_{\rm c}} [\nu_{\rm c}(|\boldsymbol{\alpha}|)\boldsymbol{\alpha} - \nu_{\rm c}(|\boldsymbol{\beta}|)\boldsymbol{\beta}] \cdot \boldsymbol{\eta} = \int_{\Omega_{\rm c}} [\boldsymbol{f}(\boldsymbol{\alpha}) - \boldsymbol{f}(\boldsymbol{\beta})] \cdot \boldsymbol{\eta}, \tag{4.41}$$

where $\boldsymbol{\alpha} := \mathbf{curl}\,\boldsymbol{u}, \boldsymbol{\beta} := \mathbf{curl}\,\boldsymbol{v}, \boldsymbol{\eta} := \mathbf{curl}\,\boldsymbol{w}$ and $\boldsymbol{f} : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$\boldsymbol{f}(\boldsymbol{x}) := \nu_{\rm c}(|\boldsymbol{x}|)\boldsymbol{x} \qquad \forall \boldsymbol{x} \in \mathbb{R}^3.$$

Let $\boldsymbol{g}(t) := \boldsymbol{f}(t\boldsymbol{\alpha} + (1-t)\boldsymbol{\beta})$. Then

$$\boldsymbol{f}(\boldsymbol{\alpha}) - \boldsymbol{f}(\boldsymbol{\beta}) = \boldsymbol{g}(1) - \boldsymbol{g}(0) = \int_0^1 \boldsymbol{g}'(t)dt = \int_0^1 \boldsymbol{Df}(t\boldsymbol{\alpha} + (1-t)\boldsymbol{\beta})(\boldsymbol{\alpha} - \boldsymbol{\beta})dt,$$

and

$$|\boldsymbol{f}(\boldsymbol{\alpha}) - \boldsymbol{f}(\boldsymbol{\beta})| \leq |\boldsymbol{\alpha} - \boldsymbol{\beta}| \int_0^1 \|\boldsymbol{Df}(t\boldsymbol{\alpha} + (1-t)\boldsymbol{\beta})\| dt,$$

with

$$\|\boldsymbol{Df}(\boldsymbol{x})\|^2 = \sum_{i,j=1}^{3} \left|\frac{\partial f_i(\boldsymbol{x})}{\partial x_j}\right|^2.$$

In order to estimate the last norm, we notice that

$$\begin{aligned}
\left|\frac{\partial f_i(\boldsymbol{x})}{\partial x_j}\right|^2 &= \left[\nu_{\rm c}'(|\boldsymbol{x}|)\frac{x_j}{|\boldsymbol{x}|}x_i + \nu_{\rm c}(|\boldsymbol{x}|)\delta_{ij}\right]^2 \\
&= [\nu_{\rm c}'(|\boldsymbol{x}|)]^2 \frac{|x_j|^2}{|\boldsymbol{x}|^2}|x_i|^2 + 2\nu_{\rm c}'(|\boldsymbol{x}|)\nu_{\rm c}(|\boldsymbol{x}|)\frac{x_j}{|\boldsymbol{x}|}x_i\delta_{ij} + [\nu_{\rm c}(|\boldsymbol{x}|)]^2\,\delta_{ij}^2 \\
&\leq [\nu_{\rm c}'(|\boldsymbol{x}|)]^2\,|\boldsymbol{x}|^2 + 2\nu_{\rm c}'(|\boldsymbol{x}|)\nu_{\rm c}(|\boldsymbol{x}|)|\boldsymbol{x}| + [\nu_{\rm c}(|\boldsymbol{x}|)]^2 \\
&\leq [\nu_{\rm c}'(|\boldsymbol{x}|)|\boldsymbol{x}| + \nu_{\rm c}(|\boldsymbol{x}|)]^2 \leq \delta_1^2,
\end{aligned}$$

where δ_1 is the Lipschitz constant of the function $s \mapsto \nu_{\rm c}(s)s$. Hence,

$$\|\boldsymbol{Df}(\boldsymbol{x})\| \leq 3\delta_1 \qquad \forall \boldsymbol{x} \neq \boldsymbol{0}. \tag{4.42}$$

Furthermore

$$\left|\frac{\partial f_i}{\partial x_j}(\boldsymbol{0})\right| = \left|\lim_{h\to 0}\frac{\nu_{\rm c}(|h|)h\delta_{i,j}}{h}\right| = \nu_{\rm c}(0)\delta_{i,j} \leq \nu_0,$$

which implies that

$$\|\boldsymbol{Df}(\boldsymbol{0})\| \leq 3\nu_0.$$

Consequently,

$$|\boldsymbol{f}(\boldsymbol{\alpha}) - \boldsymbol{f}(\boldsymbol{\beta})| \leq 3\max\{\nu_0, \delta_1\}|\boldsymbol{\alpha} - \boldsymbol{\beta}|.$$

Then, by (4.41) we obtain (4.40).

Finally, (4.39) follows by combining (4.19) and (4.40). □

Consider the linear projection operator $\Pi_h : \mathbf{H}_0(\mathbf{curl}, \Omega) \to V_{0,h}(\Omega)$ defined by (3.40).

Lemma 4.4.2 *Let $\boldsymbol{\rho}_h(t) := \boldsymbol{u}(t) - \Pi_h \boldsymbol{u}(t)$ and $\boldsymbol{\delta}_h(t) := \Pi_h \boldsymbol{u}(t) - \boldsymbol{u}_h(t)$. There exists a constant $C > 0$, independent of h, such that*

$$\begin{aligned}\sup_{t\in[0,T]} \|\boldsymbol{\delta}_h(t)\|_\sigma^2 &+ \int_0^T \|\boldsymbol{\delta}_h(t)\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2 \, dt \\ &\leq C \left\{ \int_0^T \|\partial_t \boldsymbol{\rho}_h(t)\|_\sigma^2 \, dt + \int_0^T \|\,\mathbf{curl}\, \boldsymbol{\rho}_h(t)\|_{0,\Omega}^2 \, dt \right\}.\end{aligned} \tag{4.43}$$

Proof. For any $\boldsymbol{v} \in V_{0,h}(\Omega)$ we have

$$(\partial_t \boldsymbol{\delta}_h(t), \boldsymbol{v})_\sigma + \langle A\Pi_h \boldsymbol{u}(t) - A\boldsymbol{u}_h(t), \boldsymbol{v}\rangle = -(\partial_t \boldsymbol{\rho}_h(t), \boldsymbol{v})_\sigma - \langle A\boldsymbol{u}(t) - A\Pi_h \boldsymbol{u}(t), \boldsymbol{v}\rangle.$$

By taking $\boldsymbol{v} = \boldsymbol{\delta}_h(t)$ in the last identity, using (4.38), (4.39) and the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\delta}_h(t)\|_\sigma^2 &+ \alpha \|\,\mathbf{curl}\, \boldsymbol{\delta}_h(t)\|_{0,\Omega}^2 \\ &\leq \frac{1}{4T}\|\boldsymbol{\delta}_h(t)\|_\sigma^2 + C_1 \left[\|\partial_t \boldsymbol{\rho}_h(t)\|_\sigma^2 + \|\,\mathbf{curl}\, \boldsymbol{\rho}_h(t)\|_{0,\Omega}^2\right].\end{aligned}$$

We now integrate over $[0, t]$ (note that $\boldsymbol{\delta}_h(0) = \mathbf{0}$) and use Gronwall's inequality to obtain

$$\|\boldsymbol{\delta}_h(t)\|_\sigma^2 + \int_0^t \|\,\mathbf{curl}\, \boldsymbol{\delta}_h(s)\|_{0,\Omega}^2 \, ds \leq C \int_0^T \left[\|\partial_t \boldsymbol{\rho}_h(t)\|_\sigma^2 + \|\,\mathbf{curl}\, \boldsymbol{\rho}_h(t)\|_{0,\Omega}^2\right] dt.$$

Then, taking supremum over $t \in [0, T]$ and using (3.37) we conclude (4.43). □

Theorem 4.4.2 *Assume that $\boldsymbol{u} \in \mathrm{H}^1(0, T; \mathbf{H}(\mathbf{curl}; \Omega))$ and let $\boldsymbol{e}_h(t) := \boldsymbol{u}(t) - \boldsymbol{u}_h(t)$. There exists $C > 0$, independent of h, such that*

$$\begin{aligned}\sup_{t\in[0,T]} \|\boldsymbol{e}_h(t)\|_\sigma^2 &+ \int_0^T \|\boldsymbol{e}_h(t)\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2 \, dt \\ \leq C \Bigg\{ &\int_0^T \left[\inf_{\boldsymbol{v}\in X_h(\Omega)} \|\boldsymbol{u}(t) - \boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2 + \inf_{\boldsymbol{v}\in X_h(\Omega)} \|\partial_t \boldsymbol{u}(t) - \boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2 \right] dt \\ &+ \sup_{t\in[0,T]} \inf_{\boldsymbol{v}\in X_h(\Omega)} \|\boldsymbol{u}(t) - \boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2 \Bigg\}.\end{aligned}$$

Proof. The regularity assumption on $\boldsymbol{u}$ allows us to commute the time derivative and Π_h:

$$\partial_t (\Pi_h \boldsymbol{u}(t)) = \Pi_h(\partial_t \boldsymbol{u}(t)).$$

Hence, the results follows by writing $\boldsymbol{e}_h(t) = \boldsymbol{\rho}_h(t) + \boldsymbol{\delta}_h(t)$ and using Lemmas 3.5.5 and 4.4.2. □

Corollary 4.4.1 *Let $\boldsymbol{\mathcal{X}}$ be the space defined in* (3.47). *If* $\boldsymbol{u} \in \mathrm{H}^1(0,T;\boldsymbol{\mathcal{X}} \cap \mathbf{H}_0(\mathbf{curl},\Omega))$, *then*

$$\sup_{t\in[0,T]} \|\boldsymbol{e}_h(t)\|_\sigma^2 + \int_0^T \|\boldsymbol{e}_h(t)\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2\,dt \\ \leq Ch^{2l}\left\{\sup_{t\in[0,T]}\|\boldsymbol{u}(t)\|_{\boldsymbol{\mathcal{X}}}^2 + \int_0^T \|\partial_t \boldsymbol{u}(t)\|_{\boldsymbol{\mathcal{X}}}^2\,dt\right\},$$

with $l := \min\{r,m\}$.

Proof. It is a direct consequence of Theorem 4.4.2 and the interpolation error estimate (3.48). □

4.5 Analysis of a fully-discrete scheme.

We consider a uniform partition $\{t_n := n\Delta t:\ n = 0,\dots,N\}$ of $[0,T]$ with a step size $\Delta t := \frac{T}{N}$. For any finite sequence $\{\theta^n:\ n = 0,\cdots,N\}$, let

$$\bar{\partial}\theta^n := \frac{\theta^n - \theta^{n-1}}{\Delta t}, \qquad n = 1,2,\dots,N.$$

The fully-discrete version of problem (4.12) reads as follows:
Find $(\boldsymbol{u}_h^n,\lambda_h^n) \in X_h(\Omega)\times M_h(\Omega_{\mathrm{d}}),\ n = 1,\cdots,N$, such that

$$\begin{aligned} (\bar{\partial}\boldsymbol{u}_h^n, v)_\sigma + b(v,\bar{\partial}\lambda_h^n) + \langle A\boldsymbol{u}_h^n, v\rangle &= -(\boldsymbol{J}(t_n),v)_{0,\Omega} \qquad \forall v\in X_h(\Omega),\\ b(\boldsymbol{u}_h^n,\mu) &= 0 \qquad \forall \mu\in M_h(\Omega_{\mathrm{d}}),\\ \boldsymbol{u}_h^0|_{\Omega_{\mathrm{c}}} &= \mathbf{0},\\ \lambda_h^0 &= 0. \end{aligned} \tag{4.44}$$

Hence, at each iteration step we have to find $(\boldsymbol{u}_h^n,\lambda_h^n) \in X_h(\Omega)\times M_h(\Omega_{\mathrm{d}})$ such that

$$\begin{aligned} (\boldsymbol{u}_h^n,\boldsymbol{v})_\sigma + \Delta t\,\langle A\boldsymbol{u}_h^n, v\rangle + b(\boldsymbol{v},\lambda_h^n) &= F_n(\boldsymbol{v}) \qquad \forall \boldsymbol{v}\in X_h(\Omega),\\ b(\boldsymbol{u}_h^n,\mu) &= 0 \qquad \forall\mu\in M_h(\Omega_{\mathrm{d}}), \end{aligned} \tag{4.45}$$

where

$$F_n(\boldsymbol{v}) := -\Delta t(\boldsymbol{J}(t_n),\boldsymbol{v})_{0,\Omega} + (\boldsymbol{u}_h^{n-1},\boldsymbol{v})_\sigma + b(\boldsymbol{v},\lambda_h^{n-1}).$$

In order to prove the existence and uniqueness of solution $(\boldsymbol{u}_h^n,\lambda_h^n)$ of (4.45), we first notice that the nonlinear operator $\widetilde{A}: X_h(\Omega)\to X_h(\Omega)'$ given by

$$\langle \widetilde{A}\boldsymbol{u},\boldsymbol{v}\rangle := (\boldsymbol{u},\boldsymbol{v})_\sigma + \Delta t\langle A\boldsymbol{u},\boldsymbol{v}\rangle$$

is strongly monotone and Lipschitz continuous in $V_{0,h}(\Omega)$, i.e. there exist constants $\tilde{\kappa} > 0$ and $\tilde{\alpha}$ such that

$$\langle \widetilde{A}\boldsymbol{v} - \widetilde{A}\boldsymbol{w}, \boldsymbol{v} - \boldsymbol{w}\rangle \geq \tilde{\alpha}\|\boldsymbol{v} - \boldsymbol{w}\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} \quad \forall \boldsymbol{v}, \boldsymbol{w} \in V_{0,h}(\Omega)$$

and

$$|\langle \widetilde{A}\boldsymbol{u} - \widetilde{A}\boldsymbol{v}, \boldsymbol{w}\rangle| \leq \tilde{\kappa}\|\,\mathbf{curl}\,\boldsymbol{u} - \mathbf{curl}\,\boldsymbol{v}\|_{0,\Omega}\|\,\mathbf{curl}\,\boldsymbol{w}\|_{0,\Omega} \quad \forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V_{0,h}(\Omega).$$

Consequently (see for instance [74, Theorem 25.B]), there exists a unique $\boldsymbol{u}_h^n \in V_{0,h}(\Omega) \subset X_h(\Omega)$ such that

$$(\boldsymbol{u}_h^n, \boldsymbol{v})_\sigma + \Delta t\,\langle A\boldsymbol{u}_h^n, v\rangle = F_n(\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in V_{0,h}(\Omega).$$

Then, the functional $\mathcal{G}_h^n \in X_h(\Omega)'$ defined by

$$\langle \mathcal{G}_h^n, \boldsymbol{v}\rangle := F_n(\boldsymbol{v}) - (\boldsymbol{u}_h^n, \boldsymbol{v})_\sigma - \langle A\boldsymbol{u}_h^n, \boldsymbol{v}\rangle$$

verifies

$$\langle \mathcal{G}_h^n, \boldsymbol{v}\rangle = 0 \qquad \forall \boldsymbol{v} \in V_{0,h}(\Omega).$$

Hence, since b satisfies the discrete inf-sup condition (3.39) we deduce there exists a unique $\lambda_h^n \in M_h(\Omega_\mathrm{d})$ satisfying

$$b(\boldsymbol{v}, \lambda_h^n) = \langle \mathcal{G}_h^n, \boldsymbol{v}\rangle \qquad \forall \boldsymbol{v} \in X_h(\Omega),$$

which implies that $(\boldsymbol{u}_h^n, \lambda_h^n)$ is the unique solution of (4.45).

On the other hand, testing the first equation of (4.44) with $\widetilde{\mathbf{grad}}\,\lambda_h^n$ and taking into account (4.15) leads to

$$\varepsilon_0|\lambda_h^n|^2_{1,\Omega_\mathrm{d}} = b(\mathbf{grad}\,\lambda_h^n, \lambda_h^n) = b(\mathbf{grad}\,\lambda_h^n, \lambda_h^{n-1}), \qquad n = 1, \ldots, N.$$

Consequently, the condition $\lambda_h^0 = 0$ implies that

$$\lambda_h^n = 0, \qquad n = 1, \ldots, N.$$

4.5.1 Error estimates

Lemma 4.5.1 *Let* $\boldsymbol{\rho}^n := \boldsymbol{u}(t_n) - \Pi_h \boldsymbol{u}(t_n)$, $\boldsymbol{\delta}^n := \Pi_h \boldsymbol{u}(t_n) - \boldsymbol{u}_h^n$ *and* $\boldsymbol{\tau}^n := \bar{\partial}\boldsymbol{u}(t_n) - \partial_t \boldsymbol{u}(t_n)$. *There exists a constant* $C > 0$, *independent of* h *and* Δt, *such that*

$$\begin{aligned} &\|\boldsymbol{\delta}^n\|_\sigma^2 + \Delta t \sum_{k=1}^n \|\operatorname{\mathbf{curl}} \boldsymbol{\delta}^k\|_{0,\Omega}^2 \\ &\qquad \leq C\Delta t \left(\sum_{k=1}^N \|\bar{\partial}\boldsymbol{\rho}^k\|_\sigma^2 + \sum_{k=1}^N \|\operatorname{\mathbf{curl}} \boldsymbol{\rho}^k\|_{0,\Omega}^2 + \sum_{k=1}^N \|\boldsymbol{\tau}^k\|_\sigma^2 \right), \end{aligned} \tag{4.46}$$

for all $n = 1, \ldots, N$.

Proof. It is straightforward to show that

$$\begin{aligned} &(\bar{\partial}\boldsymbol{\delta}^k, \boldsymbol{v})_\sigma + \langle A\Pi_h \boldsymbol{u}(t_k) - A\boldsymbol{u}_h^k, \boldsymbol{v}\rangle \\ &\quad = -(\bar{\partial}\boldsymbol{\rho}^k, \boldsymbol{v})_\sigma - \langle A\boldsymbol{u}(t_k) - A\Pi_h \boldsymbol{u}(t_k), \boldsymbol{v}\rangle + (\boldsymbol{\tau}^k, \boldsymbol{v})_\sigma \qquad \forall \boldsymbol{v} \in V_{0,h}. \end{aligned} \tag{4.47}$$

Choosing $\boldsymbol{v} = \boldsymbol{\delta}^k$ in the last identity and using Lemma 4.4.1, the estimate

$$(\bar{\partial}\boldsymbol{\delta}^k, \boldsymbol{\delta}^k)_\sigma \geq \frac{1}{2\Delta t} \left(\|\boldsymbol{\delta}^k\|_\sigma^2 - \|\boldsymbol{\delta}^{k-1}\|_\sigma^2 \right),$$

together with the Cauchy-Schwartz inequality, yield

$$\begin{aligned} &\|\boldsymbol{\delta}^k\|_\sigma^2 - \|\boldsymbol{\delta}^{k-1}\|_\sigma^2 + \Delta t\, \alpha \|\operatorname{\mathbf{curl}} \boldsymbol{\delta}^k\|_{0,\Omega}^2 \\ &\qquad \leq \frac{\Delta t}{2T} \|\boldsymbol{\delta}^k\|_\sigma^2 + C_1 \Delta t \left(\|\bar{\partial}\boldsymbol{\rho}^k\|_\sigma^2 + \|\operatorname{\mathbf{curl}} \boldsymbol{\rho}^k\|_{0,\Omega} + \|\boldsymbol{\tau}^k\|_\sigma^2 \right). \end{aligned} \tag{4.48}$$

In particular,

$$\|\boldsymbol{\delta}^k\|_\sigma^2 - \|\boldsymbol{\delta}^{k-1}\|_\sigma^2 \leq \frac{\Delta t}{2T} \|\boldsymbol{\delta}^k\|_\sigma^2 + C_1 \Delta t \left(\|\bar{\partial}\boldsymbol{\rho}^k\|_\sigma^2 + \|\operatorname{\mathbf{curl}} \boldsymbol{\rho}^k\|_{0,\Omega} + \|\boldsymbol{\tau}^k\|_\sigma^2 \right).$$

Then, summing over k and using the discrete Gronwall's Lemma (see, for instance, [62, Lemma 1.4.2]) lead to

$$\|\boldsymbol{\delta}^n\|_\sigma^2 \leq C_2 \Delta t \sum_{k=1}^n \left(\|\bar{\partial}\boldsymbol{\rho}^k\|_\sigma^2 + \|\operatorname{\mathbf{curl}} \boldsymbol{\rho}^k\|_{0,\Omega}^2 + \|\boldsymbol{\tau}^k\|_\sigma^2 \right),$$

for $n = 1, \ldots, N$. Inserting the last inequality in (4.48) and summing over k we have the estimate (4.46). $\square$

Theorem 4.5.1 *Assume that* $\boldsymbol{u} \in \mathrm{H}^2(0,T;\mathbf{H}(\mathbf{curl};\Omega))$ *and let* $\boldsymbol{e}^n := \boldsymbol{u}(t_n) - \boldsymbol{u}_h^n$. *Then, there exists a constant* $C > 0$, *independent of* h *and* Δt, *such that*

$$\max_{1\le n\le N} \|\boldsymbol{e}^n\|_\sigma^2 + \Delta t \sum_{k=1}^N \|\boldsymbol{e}^k\|^2_{\mathbf{H}(\mathbf{curl};\Omega)}$$
$$\le C\left\{ \max_{1\le n\le N} \inf_{\boldsymbol{v}\in X_h(\Omega)} \|\boldsymbol{u}(t_n) - \boldsymbol{v}\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} + \Delta t \sum_{n=1}^N \inf_{\boldsymbol{v}\in X_h(\Omega)} \|\boldsymbol{u}(t_n) - \boldsymbol{v}\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} \right.$$
$$\left. + \int_0^T \left(\inf_{\boldsymbol{v}\in X_h(\Omega)} \|\partial_t \boldsymbol{u}(t) - \boldsymbol{v}\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} \right) dt + \Delta t^2 \int_0^T \|\partial_{tt}\boldsymbol{u}(t)\|_\sigma^2 \, dt \right\}.$$

Proof. A Taylor expansion shows that

$$\sum_{k=1}^n \|\boldsymbol{\tau}^k\|_\sigma^2 = \sum_{k=1}^n \left\| \frac{1}{\Delta t} \int_{t_{k-1}}^{t_k} (t_{k-1} - t)\partial_{tt}\boldsymbol{u}(t)\, dt \right\|_\sigma^2 \le \Delta t \int_0^T \|\partial_{tt}\boldsymbol{u}(t)\|_\sigma^2 \, dt. \tag{4.49}$$

Moreover,

$$\sum_{k=1}^n \|\bar\partial \boldsymbol{\rho}^k\|_\sigma^2 \le \frac{1}{\Delta t} \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \|\partial_t \boldsymbol{\rho}_h(t)\|_\sigma^2 \, dt \le \frac{1}{\Delta t} \int_0^T \|\partial_t \boldsymbol{\rho}_h(t)\|_\sigma^2 \, dt. \tag{4.50}$$

Combining (4.46), (4.49), and (4.50) and recalling that $\|\cdot\|_{\mathbf{H}(\mathbf{curl};\Omega)}$ is equivalent to $\|\cdot\|_{V_0(\Omega)}$ in $V_{0,h}(\Omega)$, we obtain

$$\max_{1\le n\le N} \|\boldsymbol{\delta}^n\|_\sigma^2 + \Delta t \sum_{k=1}^N \|\boldsymbol{\delta}^k\|^2_{\mathbf{H}(\mathbf{curl};\Omega)}$$
$$\le C_0 \left\{ \int_0^T \|\partial_t \boldsymbol{\rho}_h(t)\|_\sigma^2 \, dt + \Delta t \sum_{k=1}^N \|\mathbf{curl}\, \boldsymbol{\rho}_h(t_k)\|^2_{0,\Omega} + (\Delta t)^2 \int_0^T \|\partial_{tt}\boldsymbol{u}(s)\|_\sigma^2 \, ds \right\}.$$

The result follows from the fact that $\boldsymbol{e}^n = \boldsymbol{\delta}^n + \boldsymbol{\rho}^n$ and the triangle inequality. □

Finally we deduce from (3.48) the following asymptotic error estimate.

Corollary 4.5.1 *Under the assumptions of Corollary 4.4.1 and Theorem 4.5.1, there exists a constant* C, *independent of* h *and* Δt, *such that*

$$\max_{1\le n\le N} \|\boldsymbol{e}^n\|_\sigma^2 + \Delta t \sum_{k=1}^N \|\boldsymbol{e}^k\|^2_{\mathbf{H}(\mathbf{curl};\Omega)}$$
$$\le C h^{2l} \left\{ \max_{1\le n\le N} \|\boldsymbol{u}(t_n)\|^2_{\boldsymbol{\mathcal{X}}} + \int_0^T \|\partial_t \boldsymbol{u}(t)\|^2_{\boldsymbol{\mathcal{X}}} \, dt \right\}$$
$$+ C(\Delta t)^2 \int_0^T \|\partial_{tt}\boldsymbol{u}(t)\|_\sigma^2 \, dt,$$

with $l := \min\{m, r\}$.

Chapter 5

A mixed-FEM and BEM coupling for a time-dependent eddy current problem

5.1 Introduction

The eddy current problem is naturally formulated in the whole space with decay conditions on the fields at infinity (see, for instance, [13]). Consequently, to apply conventional numerical methods, such as the finite element method (FEM), it is necessary to reduce the problem to a bounded domain. The most common approach consists in restricting the equations to a sufficiently large computational domain containing the region of interest and imposing an artificial homogeneous boundary condition on its border (which must be "sufficiently" far away from the conductor). This strategy yields the difficulty of fixing a convenient cut-off distance a priori. Moreover, in case of conductors with a "special" shape or a very large computational domain, a finite element mesh can lead to a very large number of elements. On the other hand, methods based on boundary integral equations, like the boundary element method (BEM), in general can not be directly applied because the equations are not homogeneous and have variable coefficients.

Since the equations of the eddy current problem are complex only in a bounded region, techniques combining BEM and FEM look convenient. The first FEM-BEM couplings for the eddy current model have been proposed by engineers: Bossavit and Vérité [29, 30] (using the magnetic field $\boldsymbol{H}$ in the conductor and the Steklov-Poincaré operator) and

Mayergoyz, Chari and Konrad [54] (using the electric field $\boldsymbol{E}$ in the conductor and certain harmonic basis functions near its boundary Σ). From a mathematical point of view, more recent results based on the well-known symmetric method of Costabel [37] are due to Hiptmair [46] (using $\boldsymbol{E}$ in the conductor and $\boldsymbol{H} \times \boldsymbol{n}$ on Σ) and Meddahi and Selgas [55] (using $\boldsymbol{H}$ in the conductor and the normal trace of the magnetic induction on Σ) for the time-harmonic problem. Another FEM-BEM approach for the same problem in terms of vector and scalar potentials has been also recently analyzed by Alonso and Valli [12].

When the conductor is multiply-connected, the approach mentioned above require the construction of cumbersome (and expensive) cutting surfaces in order to deal correctly with the discrete problem, see also [16, 75]. To tackle with this difficulty, Meddahi and Selgas [56] have proposed a mixed-FEM and BEM coupling for the time-dependent eddy current problem. In this work, the authors introduce a simply connected artificial computational domain Ω (with a connected boundary Γ) containing the conductor and the support of the current density. They use the magnetic field $\boldsymbol{H}$ as main variable in Ω and introduce a Lagrange multiplier to handle the constraint of $\boldsymbol{H}$ in the dielectric medium (see [8] for a similar approach), while the boundary variable on Γ is the exterior normal trace of $\boldsymbol{H}$. Another mixed-FEM and BEM formulations have been studied in [39, 50, 53] for the static eddy current problem.

The goal of this chapter is to introduce a new method to solve the time-dependent eddy current problem, based on a mixed-FEM and BEM coupling. We follow the strategy used in [56] to avoid the difficulty arising from the topology of the conductor and, as usual, the fields effect in the complementary unbounded domain, is computed through a suitable integral representation formula that provide non-local boundary conditions. We use as main variable a time primitive of $\boldsymbol{E}$ in Ω (see also [28]), a Lagrange multiplier to impose the free-divergence condition in the insulating material and a certain scalar potential on the interface boundary. This approach extends our previous work [1], where the eddy current problem reduced to a bounded domain is studied. We deduce a suitable symmetric variational formulation and its well-posedness is proved by reducing the problem to a saddle-point formulation in Ω and using similar techniques to the ones given in [1].

A feature of our formulation is that the compact support of the current density is not necessarily assumed to be completely contained in the conductor or in its exterior. Furthermore, the choice of Ω simply connected with a connected boundary, allow us to take the boundary variable in $\mathrm{H}^{1/2}(\Gamma)$, which can be approximated by means of standard nodal finite elements. On the other hand, in contrast with the formulation given in [56], our approach fits well into the theory of monotone operators, because the reluctivity (the inverse of the magnetic permeability) appears as a diffusion coefficient in the degenerate parabolic problem at hand (see (5.14) below). Consequently, this approach seems convenient when the relation between the magnetic field and the magnetic induction (given by the reluctivity) depends on the magnetic induction intensity, which is typical for the ferromagnetic materials.

We perform a space discretization of our weak formulation by using Nédélec edge elements for the main unknown and standard finite elements for the Lagrange multiplier and the boundary variable. We show that our semi-discrete Galerkin scheme is uniquely solvable and provide error estimates in terms of the space discretization parameter h. We also propose a fully-discrete Galerkin scheme based on a backward-Euler time-stepping. Here again we provide error estimates that prove optimal convergence. Moreover, we obtain error estimates for the eddy currents and the magnetic induction field.

The chapter is organized as follows. In Section 5.2, we summarize some results from [33, 31, 34] concerning tangential differential operators and traces in $\mathbf{H}(\mathbf{curl};\Omega)$ and recall some basic results for the study of time-dependent problems. In Section 5.3, we introduce the model problem. Next, we deduce a symmetric coupling variational formulation in Section 5.4 and prove that it is uniquely solvable in Section 5.5. The derivation of a semi-discretization in space and its convergence analysis are reported in Section 5.6. Finally, a backward Euler method is employed to obtain a time discretization of the problem. The results presented in Section 5.6 prove that the resulting fully discrete scheme is convergent in an optimal way.

5.2 Preliminaries

We use boldface letters to denote vectors as well as vector–valued functions and the symbol $|\cdot|$ represents the standard Euclidean norm for vectors. In this section Ω is a generic Lipschitz bounded domain of $\mathbb{R}^3$. We denote by Γ its boundary and by $\boldsymbol{n}$ the unit outward normal to Ω. Let

$$(f, g)_{0,\Omega} := \int_\Omega f g$$

be the inner product in $\mathrm{L}^2(\Omega)$ and $\|\cdot\|_{0,\Omega}$ the corresponding norm. As usual, for all $s > 0$, $\|\cdot\|_{s,\Omega}$ stands for the norm of the Hilbertian Sobolev space $\mathrm{H}^s(\Omega)$ and $|\cdot|_{s,\Omega}$ for the corresponding seminorm. The space $\mathrm{H}^{1/2}(\Gamma)$ is defined by localization on the Lipschitz surface Γ. We denote by $\|\cdot\|_{1/2,\Gamma}$ the norm in $\mathrm{H}^{1/2}(\Gamma)$ and $\langle\cdot,\cdot\rangle_{1/2,\Gamma}$ stands for the duality pairing between $\mathrm{H}^{1/2}(\Gamma)$ and its dual $\mathrm{H}^{-1/2}(\Gamma)$. From now on we denote by $\gamma : \mathrm{H}^1(\Omega) \to \mathrm{H}^{1/2}(\Gamma)$ and $\boldsymbol{\gamma} : \mathrm{H}^1(\Omega)^3 \to \mathrm{H}^{1/2}(\Gamma)^3$ the standard trace operator acting on scalar and vector fields respectively.

5.2.1 Tangential differential operators and traces

We consider the space

$$\mathbf{L}^2_\tau(\Gamma) := \left\{\boldsymbol{\lambda} \in \mathrm{L}^2(\Gamma)^3 :\ \boldsymbol{\lambda}\cdot\boldsymbol{n} = 0\right\},$$

endowed with the standard norm in $\mathrm{L}^2(\Gamma)^3$. We define the tangential trace $\boldsymbol{\gamma}_\tau : \mathcal{C}^\infty(\overline{\Omega})^3 \to \mathbf{L}^2_\tau(\Gamma)$ and the tangential component trace $\boldsymbol{\pi}_\tau : C^\infty\big(\overline{\Omega}\big)^3 \to \mathbf{L}^2_\tau(\Gamma)$ as $\boldsymbol{\gamma}_\tau\boldsymbol{v} := \boldsymbol{\gamma}\boldsymbol{v}\times\boldsymbol{n}$ and $\boldsymbol{\pi}_\tau\boldsymbol{v} := \boldsymbol{n}\times(\boldsymbol{\gamma}\boldsymbol{v}\times\boldsymbol{n})$ respectively. The previous traces can be extended by completeness to $\mathrm{H}^1(\Omega)^3$. The spaces $\mathbf{H}^{1/2}_\perp(\Gamma) := \boldsymbol{\gamma}_\tau(\mathrm{H}^1(\Omega)^3)$ and $\mathbf{H}^{1/2}_\parallel(\Gamma) := \boldsymbol{\pi}_\tau\,(H^1(\Omega)^3)$, are respectively endowed with the Hilbert norms

$$\|\boldsymbol{\eta}\|_{\mathbf{H}^{1/2}_\perp(\Gamma)} := \inf_{\boldsymbol{w}\in\mathrm{H}^1(\Omega)^3}\left\{\|\boldsymbol{w}\|_{1,\Omega} :\ \boldsymbol{\gamma}_\tau\boldsymbol{w} = \boldsymbol{\eta}\right\},$$
$$\|\boldsymbol{\eta}\|_{\mathbf{H}^{1/2}_\parallel(\Gamma)} := \inf_{\boldsymbol{w}\in\mathrm{H}^1(\Omega)^3}\left\{\|\boldsymbol{w}\|_{1,\Omega} :\ \boldsymbol{\pi}_\tau\boldsymbol{w} = \boldsymbol{\eta}\right\}.$$

Let us notice that the density of $\mathrm{H}^{1/2}(\Gamma)^3$ in $\mathrm{L}^2(\Gamma)^3$ ensures that $\mathbf{H}^{1/2}_\perp(\Gamma)$ and $\mathbf{H}^{1/2}_\parallel(\Gamma)$ are dense subspaces of $\mathbf{L}^2_\tau(\Gamma)$. We denote by $\mathbf{H}^{-1/2}_\perp(\Gamma)$ and $\mathbf{H}^{-1/2}_\parallel(\Gamma)$ the dual spaces of $\mathbf{H}^{1/2}_\perp(\Gamma)$ and $\mathbf{H}^{1/2}_\parallel(\Gamma)$ with $\mathbf{L}^2_\tau(\Gamma)$ as pivot space, with duality pairing $\langle\cdot,\cdot\rangle_{\perp,\Gamma}$ and $\langle\cdot,\cdot\rangle_{\parallel,\Gamma}$ respectively.

We introduce the tangential differential operators

$$\mathbf{grad}_\Gamma\,\varphi := \boldsymbol{\pi}_\tau(\mathbf{grad}\,\varphi) \quad \text{and} \quad \mathbf{curl}_\Gamma\,\varphi := \boldsymbol{\gamma}_\tau(\mathbf{grad}\,\varphi) \qquad \forall\varphi\in\mathrm{H}^2(\Omega).$$

Let $\mathrm{H}^{3/2}(\Gamma) := \gamma(\mathrm{H}^2(\Omega))$. It is well known that the previous operators depend only on the trace $\gamma(\varphi)$ on Γ, which implies that

$$\mathbf{grad}_\Gamma : \mathrm{H}^{3/2}(\Gamma) \to \mathbf{H}^{1/2}_\parallel(\Gamma) \quad \text{and} \quad \mathbf{curl}_\Gamma : \mathrm{H}^{3/2}(\Gamma) \to \mathbf{H}^{1/2}_\perp(\Gamma) \tag{5.1}$$

are linear and continuous (cf. [34, Proposition 3.4]). Let $\mathrm{H}^{-3/2}(\Gamma)$ be the dual space of $\mathrm{H}^{3/2}(\Gamma)$ with $\mathrm{L}^2(\Gamma)$ as pivot space. We define

$$\mathrm{div}_\Gamma : \mathbf{H}^{-1/2}_\parallel(\Gamma) \to \mathrm{H}^{-3/2}(\Gamma) \quad \text{and} \quad \mathrm{curl}_\Gamma : \mathbf{H}^{-1/2}_\perp(\Gamma) \to \mathrm{H}^{-3/2}(\Gamma), \tag{5.2}$$

by the dualities

$$\begin{aligned} \langle \mathrm{div}_\Gamma \boldsymbol{\eta}, \phi\rangle_{3/2,\Gamma} &= -\left\langle \boldsymbol{\eta}, \mathbf{grad}_\Gamma\, \phi\right\rangle_{\parallel,\Gamma} && \forall \phi \in \mathrm{H}^{3/2}(\Gamma) \quad \forall \boldsymbol{\eta} \in \mathbf{H}_{\parallel}^{-1/2}(\Gamma), \\ \langle \mathrm{curl}_\Gamma \boldsymbol{\xi}, \phi\rangle_{3/2,\Gamma} &= \left\langle \boldsymbol{\xi}, \mathbf{curl}_\Gamma\, \phi\right\rangle_{\perp,\Gamma} && \forall \phi \in \mathrm{H}^{3/2}(\Gamma) \quad \forall \boldsymbol{\xi} \in \mathbf{H}_{\perp}^{-1/2}(\Gamma). \end{aligned} \tag{5.3}$$

The following proposition is proved in [34, Proposition 3.6].

Proposition 5.2.1 *The operators* $\mathbf{grad}_\Gamma$ *and* $\mathbf{curl}_\Gamma$ *given in* (5.1) *can be extended to* $\mathrm{H}^{1/2}(\Gamma)$*. Moreover,* $\mathbf{grad}_\Gamma : \mathrm{H}^{1/2}(\Gamma) \to \mathbf{H}_{\perp}^{-1/2}(\Gamma)$ *and* $\mathbf{curl}_\Gamma : \mathrm{H}^{1/2}(\Gamma) \to \mathbf{H}_{\parallel}^{-1/2}(\Gamma)$ *are linear and continuous. Analogously, the adjoint operators introduced in* (5.2) *are also continuous for the following choice of spaces:* $\mathrm{div}_\Gamma : \mathbf{H}_{\perp}^{1/2}(\Gamma) \to \mathrm{H}^{-1/2}(\Gamma)$ *and* $\mathrm{curl}_\Gamma : \mathbf{H}_{\parallel}^{1/2}(\Gamma) \to \mathrm{H}^{-1/2}(\Gamma)$*. Furthermore, analogous identities to* (5.3) *still hold for any* $\phi \in \mathrm{H}^{1/2}(\Gamma)$, $\boldsymbol{\eta} \in \mathbf{H}_{\perp}^{1/2}(\Gamma)$ *and* $\boldsymbol{\xi} \in \mathbf{H}_{\parallel}^{1/2}(\Gamma)$*. More precisely, we have*

$$\begin{aligned} \langle \mathrm{div}_\Gamma \boldsymbol{\eta}, \phi\rangle_{1/2,\Gamma} &= -\left\langle \mathbf{grad}_\Gamma\, \phi, \boldsymbol{\eta}\right\rangle_{\perp,\Gamma} && \forall \phi \in \mathrm{H}^{1/2}(\Gamma) \quad \forall \boldsymbol{\eta} \in \mathbf{H}_{\perp}^{1/2}(\Gamma), \\ \langle \mathrm{curl}_\Gamma \boldsymbol{\xi}, \phi\rangle_{1/2,\Gamma} &= \left\langle \mathbf{curl}_\Gamma\, \phi, \boldsymbol{\xi}\right\rangle_{\parallel,\Gamma} && \forall \phi \in \mathrm{H}^{1/2}(\Gamma) \quad \forall \boldsymbol{\xi} \in \mathbf{H}_{\parallel}^{1/2}(\Gamma). \end{aligned}$$

Let

$$\mathbf{H}(\mathbf{curl};\Omega) := \left\{ \boldsymbol{v} \in \mathrm{L}^2(\Omega)^3 :\ \mathbf{curl}\, \boldsymbol{v} \in \mathrm{L}^2(\Omega)^3 \right\},$$

endowed with the norm

$$\|\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)} := \left(\|\boldsymbol{v}\|_{0,\Omega}^2 + \|\mathbf{curl}\, \boldsymbol{v}\|_{0,\Omega}^2 \right)^{1/2}. \tag{5.4}$$

Using the Green formula (see, for instance, [31] for the case of Lipschitz polyhedra and [34] for arbitrary Lipschitz domains)

$$(\boldsymbol{u}, \mathbf{curl}\, \boldsymbol{v})_{0,\Omega} - (\mathbf{curl}\, \boldsymbol{u}, \boldsymbol{v})_{0,\Omega} = \left\langle \boldsymbol{\gamma}_\tau \boldsymbol{u}, \boldsymbol{\pi}_\tau \boldsymbol{v}\right\rangle_{\parallel,\Gamma} = -\left\langle \boldsymbol{\pi}_\tau \boldsymbol{v}, \boldsymbol{\gamma}_\tau \boldsymbol{u}\right\rangle_{\perp,\Gamma} \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathcal{C}^\infty(\overline{\Omega})^3,$$

and the density of $\mathcal{C}^\infty(\overline{\Omega})^3$ in $\mathbf{H}(\mathbf{curl}, \Omega)$ (see, for instance, [57, Theorem 3.26]) and in $\mathrm{H}^1(\Omega)$, it follows that

$$\boldsymbol{\gamma}_\tau :\ \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}_{\parallel}^{-1/2}(\Gamma), \qquad \boldsymbol{\pi}_\tau :\ \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}_{\perp}^{-1/2}(\Gamma)$$

are continuous. The ranges of these operators are characterized in the following result.

Theorem 5.2.1 *Let*

$$\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma; \Gamma) := \left\{ \boldsymbol{\lambda} \in \mathbf{H}_{\parallel}^{-1/2}(\Gamma) :\ \mathrm{div}_\Gamma \boldsymbol{\lambda} \in \mathrm{H}^{-1/2}(\Gamma) \right\}$$

and

$$\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma;\Gamma) := \left\{\boldsymbol{\lambda} \in \mathbf{H}_\perp^{-1/2}(\Gamma) :\ \mathrm{curl}_\Gamma \boldsymbol{\lambda} \in \mathrm{H}^{-1/2}(\Gamma)\right\}.$$

Then

$$\boldsymbol{\gamma}_\tau : \mathbf{H}(\mathbf{curl};\Omega) \to \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma;\Gamma), \quad \boldsymbol{\pi}_\tau : \mathbf{H}(\mathbf{curl};\Omega) \to \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma;\Gamma)$$

are surjective and possess a continuous right inverse.

The spaces $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma;\Gamma)$ *and* $\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma;\Gamma)$ *are dual to each other, when* $\mathbf{L}^2_\tau(\Gamma)$ *is used as pivot space, i.e. the usual* $\mathbf{L}^2_\tau(\Gamma)$*-inner product can be extended to a duality pairing* $\langle\cdot,\cdot\rangle_{\tau,\Gamma}$ *between* $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma;\Gamma)$ *and* $\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma;\Gamma)$*. Moreover, the following integration by parts formula holds true*

$$(\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})_{0,\Omega} - (\mathbf{curl}\,\boldsymbol{u}, \boldsymbol{v})_{0,\Omega} = \langle \boldsymbol{\gamma}_\tau \boldsymbol{u}, \boldsymbol{\pi}_\tau \boldsymbol{v}\rangle_{\tau,\Gamma} \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathbf{H}(\mathbf{curl};\Omega). \tag{5.5}$$

Proof. See Theorem 4.1 and Lemma 5.6 of [34]. □

Let Ω be a Lipschitz polyhedron. The following Theorem gives a characterization of the space

$$\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma 0;\Gamma) := \left\{\boldsymbol{\eta} \in \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma;\Gamma) :\ \mathrm{div}_\Gamma \boldsymbol{\eta} = 0\right\}.$$

Theorem 5.2.2 *Let* $\mathcal{O}$ *be a regular bounded open connected and simply connected subset of* $\mathbb{R}^3$*, such that* $\overline{\Omega} \subset \mathcal{O}$*. We set* $\Omega_{\mathrm{ext}} := \mathcal{O} \setminus \overline{\Omega}$*. Let* $\mathbb{H}_1$ *and* $\mathbb{H}_2$ *the spaces of the so-called harmonic Neumann fields associated to* Ω *and* Ω_{ext} *respectively, i.e.*

$$\mathbb{H}_1 := \{\boldsymbol{v} \in \mathbf{H}(\mathbf{curl};\Omega) \cap \mathrm{H}(\mathrm{div};\Omega) :\ \mathbf{curl}\,\boldsymbol{v} = \mathbf{0},\ \mathrm{div}\,\boldsymbol{v} = 0,\ \boldsymbol{v}\cdot\boldsymbol{n}|_\Gamma = 0\},$$

$$\mathbb{H}_2 := \{\boldsymbol{v} \in \mathbf{H}(\mathbf{curl};\Omega_{\mathrm{ext}}) \cap \mathrm{H}(\mathrm{div};\Omega_{\mathrm{ext}}) :\ \mathbf{curl}\,\boldsymbol{v} = \mathbf{0},\ \mathrm{div}\,\boldsymbol{v} = 0,\ \boldsymbol{v}\cdot\boldsymbol{n}|_{\partial\Omega_{\mathrm{ext}}} = 0\}.$$

Let $\boldsymbol{\eta} \in \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma;\Gamma)$*. Then,* $\mathrm{div}_\Gamma \boldsymbol{\eta} = 0$ *if and only if there exists* $\lambda \in \mathrm{H}^{1/2}(\Gamma)$*,* $\boldsymbol{v}_1 \in \mathbb{H}_1$ *and* $\boldsymbol{v}_2 \in \mathbb{H}_2$ *such that*

$$\boldsymbol{\eta} = \mathbf{curl}_\Gamma\,\lambda + \boldsymbol{\pi}_\tau \boldsymbol{v}_1 + \boldsymbol{\pi}_\tau \boldsymbol{v}_2|_\Gamma.$$

Proof. See [33, Section 3]. □

If Ω has a trivial topology (e.g. simply-connected), it is well know that $\mathbb{H}_1 = \mathbb{H}_2 = \{\mathbf{0}\}$ (see, for instance, [14, Subsection 3.3]). Therefore, the previous theorem implies $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma 0;\Gamma) = \mathbf{curl}_\Gamma(\mathrm{H}^{1/2}(\Gamma))$. Furthermore, if Γ is connected then $\ker(\mathbf{curl}_\Gamma) \cap \mathrm{H}^{1/2}(\Gamma) = \mathbb{R}$ (cf. [34, Corollary 3.7]). Consequently, the next result follows immediately from Proposition 5.2.1.

Corollary 5.2.1 *Let*

$$\mathrm{H}_0^{1/2}(\Gamma) := \left\{ \eta \in \mathrm{H}^{1/2}(\Gamma) : \int_\Gamma \eta = 0 \right\}.$$

If Ω is simply connected and Γ is connected, then the operator

$$\mathbf{curl}_\Gamma : \mathrm{H}_0^{1/2}(\Gamma) \to \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma 0; \Gamma)$$

is an isomorphism.

We will also use the normal trace $\gamma_{\boldsymbol{n}} : \mathcal{C}^\infty(\overline{\Omega})^3 \to \mathrm{L}^2(\Gamma)$ given by $\boldsymbol{q} \mapsto \boldsymbol{\gamma}\boldsymbol{q} \cdot \boldsymbol{n}$. It is well known that this operator can be extended to a continuous and surjective mapping (see, for instance, [57, Theorem 3.24])

$$\gamma_{\boldsymbol{n}} : \mathbf{H}(\mathrm{div}, \Omega) \to \mathrm{H}^{-1/2}(\Gamma),$$

where

$$\mathbf{H}(\mathrm{div}, \Omega) := \left\{ \boldsymbol{q} \in \mathrm{L}^2(\Omega)^3 : \mathrm{div}\, \boldsymbol{q} \in \mathrm{L}^2(\Omega) \right\}$$

is endowed with the norm

$$\|\boldsymbol{v}\|_{\mathrm{H}(\mathrm{div};\Omega)} := \left(\|\boldsymbol{v}\|_{0,\Omega}^2 + \|\mathrm{div}\, \boldsymbol{v}\|_{0,\Omega}^2 \right)^{1/2}.$$

5.2.2 Basic spaces for time dependent problems

Since we will deal with a time-domain problem, besides the Sobolev spaces defined above, we need to introduce spaces of functions defined on a bounded time interval $(0, T)$ and with values in a separable Hilbert space V, whose norm is denoted here by $\|\cdot\|_V$. We use the notation $\mathcal{C}^0([0, T]; V)$ for the Banach space consisting of all continuous functions $f : [0, T] \to V$. More generally, for any $k \in \mathbb{N}$, $\mathcal{C}^k([0, T]; V)$ denotes the subspace of $\mathcal{C}^0([0, T]; V)$ of all functions f with (strong) derivatives in $\mathcal{C}^0([0, T]; V)$, *i.e.*

$$\mathcal{C}^k([0, T]; V) := \left\{ f \in \mathcal{C}^0([0, T]; V) : \frac{d^j f}{dt^j} \in \mathcal{C}^0([0, T]; V),\ 1 \le j \le k \right\}.$$

We also consider the space $\mathrm{L}^2(0, T; V)$ of classes of functions $f : (0, T) \to V$ that are Böchner-measurable and such that

$$\|f\|_{\mathrm{L}^2(0,T;V)}^2 := \int_0^T \|f(t)\|_V^2\, dt < +\infty.$$

Furthermore, we will use the space

$$\mathrm{H}^1(0,T;V) := \left\{ f \in \mathrm{L}^2(0,T;V) : \ \frac{d}{dt} f \in \mathrm{L}^2(0,T;V) \right\},$$

where $\frac{d}{dt} f$ is the (generalized) time derivative of f; see, for instance [73, Section 23.5]. In what follows, we will use indistinctly the notations

$$\frac{d}{dt} f = \partial_t f$$

to express the time derivative of f. Analogously, we define $\mathrm{H}^k(0,T;V)$ for all $k \in \mathbb{N}$.

5.3 The model problem

We assume that the conductor is represented by a bounded Lipschitz polyhedron $\Omega_c \subset \mathbb{R}^3$. Let $\Sigma := \partial\Omega_c$ and we denote by Σ_i, $i = 1, \ldots, I$ the connected components of Σ.

Given a time-dependent compactly supported current density $\boldsymbol{J}$, our aim is to find the electric field $\boldsymbol{E}(\boldsymbol{x}, t)$ and the magnetic field $\boldsymbol{H}(\boldsymbol{x}, t)$ satisfying the following equations:

$$\partial_t (\mu \boldsymbol{H}) + \mathbf{curl}\, \boldsymbol{E} = \mathbf{0} \qquad \text{in } \mathbb{R}^3 \times (0,T), \tag{5.6}$$

$$\mathbf{curl}\, \boldsymbol{H} = \boldsymbol{J} + \sigma \boldsymbol{E} \qquad \text{in } \mathbb{R}^3 \times [0,T), \tag{5.7}$$

$$\operatorname{div}(\varepsilon \boldsymbol{E}) = 0 \qquad \text{in } \left(\mathbb{R}^3 \setminus \Omega_c\right) \times [0,T), \tag{5.8}$$

$$\int_{\Sigma_i} \varepsilon \boldsymbol{E} \cdot \boldsymbol{n} = 0 \qquad \text{in } [0,T), \quad i = 1, \cdots, I, \tag{5.9}$$

$$\boldsymbol{H}(\boldsymbol{x}, 0) = \boldsymbol{H}_0(\boldsymbol{x}) \qquad \text{in } \mathbb{R}^3, \tag{5.10}$$

$$\boldsymbol{H}(\boldsymbol{x}, t) = O\left(\frac{1}{|\boldsymbol{x}|}\right) \quad \text{and} \quad \boldsymbol{E}(\boldsymbol{x}, t) = O\left(\frac{1}{|\boldsymbol{x}|}\right) \qquad \text{as } |\boldsymbol{x}| \to \infty, \tag{5.11}$$

where the asymptotic behavior (5.11) holds uniformly in $[0,T]$. The electric permittivity ε, the electric conductivity σ, and the magnetic permeability μ are piecewise smooth real valued functions satisfying:

$$\varepsilon_1 \geq \varepsilon(\boldsymbol{x}) \geq \varepsilon_0 > 0 \quad \text{a.e. in } \Omega_c \qquad \text{and} \qquad \varepsilon(\boldsymbol{x}) = \varepsilon_0 \quad \text{a.e. in } \mathbb{R}^3 \setminus \Omega_c,$$

$$\sigma_1 \geq \sigma(\boldsymbol{x}) \geq \sigma_0 > 0 \quad \text{a.e. in } \Omega_c \qquad \text{and} \qquad \sigma(\boldsymbol{x}) = 0 \quad \text{a.e. in } \mathbb{R}^3 \setminus \Omega_c,$$

$$\mu_1 \geq \mu(\boldsymbol{x}) \geq \mu_0 > 0 \quad \text{a.e. in } \Omega_c \qquad \text{and} \qquad \mu(\boldsymbol{x}) = \mu_0 \quad \text{a.e. in } (\mathbb{R}^3 \setminus \Omega_c).$$

It is important to notice that, since $\sigma = 0$ in Ω_{d}, (5.7) implies that the data $\boldsymbol{J}$ satisfies the compatibility conditions

$$\operatorname{div} \boldsymbol{J} = 0 \text{ in } \Omega_{\mathrm{d}} \qquad \text{and} \qquad \langle \gamma_{\boldsymbol{n}} \boldsymbol{J}, 1 \rangle_{1/2,\Sigma_i} = 0, \quad i = 1, 2, \ldots, I, \tag{5.12}$$

for all $t \in (0, T)$.

Let $\Omega \subset \mathbb{R}^3$ be a connected and simply connected polyhedron, with a connected boundary $\Gamma := \partial\Omega$, and such that $\overline{\Omega}_{\mathrm{c}} \cup \operatorname{supp} \boldsymbol{J} \subset \Omega$. We denote $\Omega_{\mathrm{d}} := \Omega \setminus \overline{\Omega}_{\mathrm{c}}$ and $\Omega' := \mathbb{R}^3 \setminus \overline{\Omega}$. The unit outward normal vector to Ω is denoted by $\boldsymbol{n}$.

For reasons that will be clear later, we need to consider a modified electric field. To this end, let us consider $\psi \in \mathrm{H}^1(\Omega_{\mathrm{d}})$ such that:

$$\begin{aligned} -\operatorname{div}(\varepsilon_0 \,\mathbf{grad}\, \psi) &= 0 && \text{in } \Omega_{\mathrm{d}}, \\ \gamma_{\boldsymbol{n}}(\varepsilon_0 \,\mathbf{grad}\, \psi) &= \gamma_{\boldsymbol{n}}(\varepsilon_0 \boldsymbol{E}) && \text{on } \Gamma, \\ \gamma(\psi) &= 0 && \text{on } \Sigma. \end{aligned}$$

Next, we extend ψ to Ω' by solving the problem of finding

$$\psi_{\mathrm{ext}} \in W^1(\Omega') := \left\{ \varphi \in \mathcal{D}'(\Omega') : \frac{\varphi}{\sqrt{1 + |\boldsymbol{x}|}} \in \mathrm{L}^2(\Omega'),\, \mathbf{grad}\, \varphi \in \mathrm{L}^2(\Omega')^3 \right\}$$

satisfying

$$\begin{aligned} -\operatorname{div}(\varepsilon_0 \,\mathbf{grad}\, \psi_{\mathrm{ext}}) &= 0 && \text{in } \Omega', \\ \gamma(\psi_{\mathrm{ext}}) &= \gamma(\psi) && \text{on } \Gamma, \\ \psi_{\mathrm{ext}}(\boldsymbol{x}) &= O\left(\frac{1}{|\boldsymbol{x}|}\right) && \text{as } |\boldsymbol{x}| \to \infty. \end{aligned}$$

Then, we consider the function $\boldsymbol{G}$ defined by

$$\boldsymbol{G} := \begin{cases} \mathbf{0} & \text{in } \Omega_{\mathrm{c}}, \\ \mathbf{grad}\, \psi & \text{in } \Omega_{\mathrm{d}}, \\ \mathbf{grad}\, \psi_{\mathrm{ext}} & \text{in } \Omega'. \end{cases}$$

It is easy to see that $\boldsymbol{G} \in \mathbf{H}(\mathbf{curl}, \mathbb{R}^3)$ with $\mathbf{curl}\, \boldsymbol{G} = \mathbf{0}$ in $\mathbb{R}^3$.

Let $\boldsymbol{E}^* := \boldsymbol{E} - \boldsymbol{G}$. Notice that $\boldsymbol{E}^* = \boldsymbol{E}$ in Ω_{c}. Furthermore, it is a simple matter to show that $(\boldsymbol{E}, \boldsymbol{H})$ satisfies (5.6)-(5.11) if and only if $(\boldsymbol{E}^*, \boldsymbol{H})$ satisfies the following equations:

$$
\begin{aligned}
\partial_t\left(\mu \boldsymbol{H}\right)+\operatorname{\mathbf{curl}} \boldsymbol{E}^{*} &= \mathbf{0} && \text{in } \Omega \times (0,T), \\
\operatorname{\mathbf{curl}} \boldsymbol{H} &= \boldsymbol{J}+\sigma \boldsymbol{E}^{*} && \text{in } \Omega \times [0,T), \\
\operatorname{div}(\varepsilon_0 \boldsymbol{E}^{*}) &= 0 && \text{in } \Omega_{\mathrm{d}} \times [0,T), \\
\int_{\Sigma_i} \varepsilon_0 \boldsymbol{E}^{*} \cdot \boldsymbol{n} &= 0 && \text{in } [0,T),\ i=1,\cdots,I, \\
\gamma_{\boldsymbol{n}}^{-}(\boldsymbol{E}^{*}) &= 0 && \text{on } \Gamma \times [0,T), \\
\boldsymbol{\gamma}_{\tau}^{-}(\boldsymbol{E}^{*}) &= \boldsymbol{\gamma}_{\tau}^{+}(\boldsymbol{E}^{*}) && \text{on } \Gamma \times [0,T), \\
\boldsymbol{\gamma}_{\tau}^{-}(\boldsymbol{H}) &= \boldsymbol{\gamma}_{\tau}^{+}(\boldsymbol{H}) && \text{on } \Gamma \times [0,T), \\
\partial_t\left(\mu_0 \boldsymbol{H}\right)+\operatorname{\mathbf{curl}} \boldsymbol{E}^{*} &= \mathbf{0} && \text{in } \Omega' \times (0,T), \\
\operatorname{\mathbf{curl}} \boldsymbol{H} &= \mathbf{0} && \text{in } \Omega' \times [0,T), \\
\operatorname{div}(\varepsilon_0 \boldsymbol{E}^{*}) &= 0 && \text{in } \Omega' \times [0,T), \\
\boldsymbol{H}(\boldsymbol{x},0) &= \boldsymbol{H}_0(\boldsymbol{x}) && \text{in } \mathbb{R}^3, \\
\boldsymbol{H}(\boldsymbol{x},t) = O\left(\frac{1}{|\boldsymbol{x}|}\right) \quad \text{and} \quad \boldsymbol{E}^{*}(\boldsymbol{x},t) &= O\left(\frac{1}{|\boldsymbol{x}|}\right) && \text{as } |\boldsymbol{x}| \to \infty.
\end{aligned}
\tag{5.13}
$$

In the equations above, $\boldsymbol{\gamma}_{\tau}^{+}$ refers to the tangential trace on Γ taken from Ω' and $\boldsymbol{\gamma}_{\tau}^{-}$ ($\gamma_{\boldsymbol{n}}^{-}$) to the tangential (normal) trace taken from Ω. We adopt the same convention for any other kind of trace.

In order to obtain a suitable variational formulation for the previous problem, we proceed as in Section 3.3and introduce the variable

$$
\boldsymbol{u}(\boldsymbol{x},t) := \int_0^t \boldsymbol{E}^{*}(\boldsymbol{x},s)\, ds.
$$

Then, we use the relationship (obtained from the first two equations of Problem (5.13))

$$
\boldsymbol{H} = -\mu^{-1} \operatorname{\mathbf{curl}} \boldsymbol{u} + \boldsymbol{H}_0,
$$

to eliminate $\boldsymbol{H}$ and obtain the following problem in terms of the new variable $\boldsymbol{u}$:

Find $\boldsymbol{u} : \mathbb{R}^3 \to \mathbb{R}^3$ such that:

$$\sigma\partial_t \boldsymbol{u} + \mathbf{curl}\,\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u} = \boldsymbol{f} \qquad \text{in } \Omega \times (0,T), \tag{5.14}$$

$$\operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } \Omega_{\mathrm{d}} \times [0,T), \tag{5.15}$$

$$\int_{\Sigma_i} \varepsilon_0 \boldsymbol{u} \cdot \boldsymbol{n} = 0 \qquad \text{in } [0,T),\ i = 1,\cdots,I, \tag{5.16}$$

$$\boldsymbol{u}(\boldsymbol{x},0) = \mathbf{0} \qquad \text{in } \Omega_{\mathrm{c}}, \tag{5.17}$$

$$\gamma_{\boldsymbol{n}}^-(\boldsymbol{u}) = 0 \qquad \text{on } \Gamma \times [0,T), \tag{5.18}$$

$$\boldsymbol{\pi}_\tau^+ \boldsymbol{u} = \boldsymbol{\pi}_\tau^- \boldsymbol{u} \qquad \text{on } \Gamma \times [0,T), \tag{5.19}$$

$$\boldsymbol{\gamma}_\tau^- \left(\mu_0^{-1}\,\mathbf{curl}\,\boldsymbol{u}\right) = \boldsymbol{\gamma}_\tau^+ \left(\mu_0^{-1}\,\mathbf{curl}\,\boldsymbol{u}\right) \qquad \text{on } \Gamma \times [0,T), \tag{5.20}$$

$$\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} = \mathbf{0} \qquad \text{in } \Omega' \times [0,T), \tag{5.21}$$

$$\operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } \Omega' \times [0,T), \tag{5.22}$$

$$\boldsymbol{u}(\boldsymbol{x},t) = O\left(\frac{1}{|\boldsymbol{x}|}\right) \qquad \text{as } |\boldsymbol{x}| \to \infty, \tag{5.23}$$

$$\mathbf{curl}\,\boldsymbol{u}(\boldsymbol{x},t) = O\left(\frac{1}{|\boldsymbol{x}|}\right) \qquad \text{as } |\boldsymbol{x}| \to \infty, \tag{5.24}$$

where

$$\boldsymbol{f} := \mathbf{curl}\,\boldsymbol{H}_0 - \boldsymbol{J}.$$

Notice that, because of (5.7) $\operatorname{supp} \boldsymbol{f} \subset \Omega$.

5.4 A mixed FEM-BEM coupling variational formulation

5.4.1 The variational formulation in Ω

Let

$$M(\Omega_{\mathrm{d}}) := \left\{ q \in \mathrm{H}^1(\Omega_{\mathrm{d}}) :\ \int_{\Omega_{\mathrm{d}}} q = 0,\ \gamma q|_{\Sigma_i} = C_i,\ i = 1,\ldots,I \right\}.$$

It is well known that $|\cdot|_{1,\Omega_{\mathrm{d}}}$ is a norm in $M(\Omega_{\mathrm{d}})$ equivalent to the $\mathrm{H}^1(\Omega_{\mathrm{d}})$-norm. We denote

$$V(\Omega) := \{ \boldsymbol{v} \in \mathbf{H}(\mathbf{curl};\Omega) :\ b(\boldsymbol{v},q) = 0 \quad \forall q \in M(\Omega_{\mathrm{d}}) \}, \tag{5.25}$$

where $b(\boldsymbol{v},q) := (\varepsilon\boldsymbol{v}, \mathbf{grad}\,q)_{0,\Omega}$.

Since $\varepsilon = \varepsilon_0$ in $\mathbb{R}^3 \setminus \overline{\Omega}_{\mathrm{c}}$, it is simple to prove the following characterization of $V(\Omega)$.

Lemma 5.4.1 *There holds*

$$V(\Omega)=\Big\{\boldsymbol{v}\in\mathbf{H}(\mathbf{curl};\Omega):\ \operatorname{div}\boldsymbol{v}=0\ \textit{in}\ \Omega_{\mathrm{d}};\ \gamma_{\boldsymbol{n}}\boldsymbol{v}=0\ \textit{on}\ \Gamma;\\ \langle\gamma_{\boldsymbol{n}}\boldsymbol{v},1\rangle_{1/2,\Sigma_i}=0,\ i=1,\dots,I\Big\}.$$

Let $\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}})'$ be the dual space of $\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}})$ with respect to the pivot space

$$\mathrm{L}^2(\Omega_{\mathrm{c}},\sigma)^3:=\left\{\boldsymbol{v}:\Omega_{\mathrm{c}}\to\mathbb{R}^3\ \text{Lebesgue-measurable}:\ \int_{\Omega_{\mathrm{c}}}\sigma|\boldsymbol{v}|^2<\infty\right\}.$$

We define

$$\mathcal{W}_0:=\left\{\boldsymbol{v}\in\mathrm{L}^2(0,T;V(\Omega)):\ \boldsymbol{v}|_{\Omega_{\mathrm{c}}}\in W^1(0,T;\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}}))\right\},$$

with

$$W^1(0,T;\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}})):=\left\{\boldsymbol{v}\in\mathrm{L}^2(0,T;\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}})):\ \partial_t\boldsymbol{v}\in\mathrm{L}^2(0,T;\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}})')\right\}.$$

It is clear that (5.15), (5.16) and the previous lemma, imply $\boldsymbol{u}\in\mathcal{W}_0$. Moreover, testing (5.14) by functions $\boldsymbol{v}\in V(\Omega)$ and using the Green formula (5.5), we obtain that $\boldsymbol{u}$ verifies:

$$\frac{d}{dt}(\sigma\boldsymbol{u}(t),\boldsymbol{v})_{0,\Omega_{\mathrm{c}}}+\left(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}(t),\mathbf{curl}\,\boldsymbol{v}\right)_{0,\Omega}\\ -\left\langle\boldsymbol{\gamma}_\tau\left(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}(t)\right),\boldsymbol{\pi}_\tau\boldsymbol{v}\right\rangle_{\tau,\Gamma}=(\boldsymbol{f}(t),\boldsymbol{v})_{0,\Omega}\qquad\forall\boldsymbol{v}\in V(\Omega).$$

Consequently, if we relax the divergence-free restriction in the previous equation (implicit in the definition of $V(\Omega)$) by introducing a Lagrange multiplier $p(t)\in M(\Omega_{\mathrm{d}})$, we obtain the following mixed formulation of the problem in Ω:
Find $\boldsymbol{u}\in\mathcal{W}$, $p\in\mathrm{L}^2(0,T;M(\Omega_{\mathrm{d}}))$ such that

$$\begin{aligned}&\frac{d}{dt}\left[(\boldsymbol{u}(t),\boldsymbol{v})_\sigma+b(\boldsymbol{v},p(t))\right]+\left(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u},\mathbf{curl}\,\boldsymbol{v}\right)_{0,\Omega}\\ &\qquad-\left\langle\boldsymbol{\gamma}_\tau^-\left(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}(t)\right),\boldsymbol{\pi}_\tau\boldsymbol{v}\right\rangle_{\tau,\Gamma}=(\boldsymbol{f}(t),\boldsymbol{v})_{0,\Omega}\quad\forall\boldsymbol{v}\in\mathbf{H}(\mathbf{curl};\Omega),\\ &\qquad b(\boldsymbol{u}(t),q)=0\qquad\forall q\in M(\Omega_{\mathrm{d}}),\\ &\qquad \boldsymbol{u}|_{\Omega_{\mathrm{c}}}(0)=\mathbf{0},\end{aligned}\tag{5.26}$$

where

$$\mathcal{W}:=\left\{\boldsymbol{v}\in\mathrm{L}^2(0,T;\mathbf{H}_0(\mathbf{curl},\Omega)):\ \boldsymbol{v}|_{\Omega_{\mathrm{c}}}\in W^1(0,T;\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}}))\right\}.$$

Notice that $\mathcal{W}$, endowed with the graph norm

$$\|\boldsymbol{v}\|_{\mathcal{W}}^2:=\int_0^T\|\boldsymbol{v}(t)\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2\,dt+\int_0^T\|\partial_t\boldsymbol{v}(t)\|_{\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}})'}^2\,dt,$$

is a Hilbert space and that $\mathcal{W}_0$ is a closed subspace of $\mathcal{W}$.

5.4.2 The symmetric FEM-BEM coupling.

Notice that, for any $q \in M(\Omega_{\mathrm{d}})$, the extension $\widetilde{\mathbf{grad}}\, q$ of $\mathbf{grad}\, q$ by zero to the whole Ω, belongs to $V(\Omega)$. Testing the first equation of (5.26) with $\boldsymbol{v} = \widetilde{\mathbf{grad}}\, q$ and taking into account that $\boldsymbol{f}$ satisfies the compatibility conditions

$$\operatorname{div} \boldsymbol{f} = 0 \text{ in } \Omega_{\mathrm{d}} \qquad \text{and} \qquad \langle \gamma_{\boldsymbol{n}} \boldsymbol{f}, 1 \rangle_{1/2, \Sigma_i} = 0, \quad i = 1, 2, \ldots, I, \tag{5.27}$$

we deduce that $\operatorname{div}_\Gamma [\boldsymbol{\gamma}_\tau^- (\mu^{-1} \mathbf{curl}\, \boldsymbol{u})] = 0$. Then, by virtue of Corollary 5.2.1, we have that for any $t \in [0, T]$ there exists a unique $\lambda(t) \in \mathrm{H}_0^{1/2}(\Gamma)$ such that

$$\boldsymbol{\gamma}_\tau^- \left(\mu^{-1} \mathbf{curl}\, \boldsymbol{u} \right) = \mathbf{curl}_\Gamma\, \lambda. \tag{5.28}$$

As the solution $\boldsymbol{u}$ verifies (5.21)-(5.24), it has the following integral representation in Ω' (see, for instance, [46, Section 5]):

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{x}) = & \mathbf{curl}_{\boldsymbol{x}} \int_\Gamma E(\boldsymbol{x}, \boldsymbol{y})\, \boldsymbol{n} \times \boldsymbol{\pi}_\tau^+ \boldsymbol{u}\, ds_{\boldsymbol{y}} - \int_\Gamma E(\boldsymbol{x}, \boldsymbol{y})\, \boldsymbol{\gamma}_\tau^+ \left(\mathbf{curl}\, \boldsymbol{u} \right) ds_{\boldsymbol{y}} \\ & - \mathbf{grad}_{\boldsymbol{x}} \int_\Gamma E(\boldsymbol{x}, \boldsymbol{y})\, \gamma_{\boldsymbol{n}}^+ \boldsymbol{u}\, ds_{\boldsymbol{y}} \qquad\qquad \forall \boldsymbol{x} \in \Omega', \end{aligned} \tag{5.29}$$

where E is the fundamental solution of the Laplace equation in $\mathbb{R}^3$, i.e.

$$E(\boldsymbol{x}, \boldsymbol{y}) := \frac{1}{4\pi |\boldsymbol{x} - \boldsymbol{y}|}, \qquad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3,\ \boldsymbol{x} \neq \boldsymbol{y}.$$

To take advantage of the previous representation formula, we consider the following boundary integral operators defined for smooth functions ϕ and $\boldsymbol{\eta}$ by:

$$\begin{aligned} S\phi(\boldsymbol{x}) &:= \gamma \left(\boldsymbol{x} \mapsto \int_\Gamma E(\boldsymbol{x}, \boldsymbol{y}) \phi(y)\, ds_{\boldsymbol{y}} \right), \\ \boldsymbol{V\eta}(\boldsymbol{x}) &:= \boldsymbol{\pi}_\tau \left(\boldsymbol{x} \mapsto \int_\Gamma E(\boldsymbol{x}, \boldsymbol{y}) \boldsymbol{\eta}(y)\, ds_{\boldsymbol{y}} \right), \\ \boldsymbol{K\eta}(\boldsymbol{x}) &:= \boldsymbol{\gamma}_\tau^+ \left(\boldsymbol{x} \mapsto \mathbf{curl}_{\boldsymbol{x}} \int_\Gamma E(\boldsymbol{x}, \boldsymbol{y}) \boldsymbol{\eta}(y)\, ds_{\boldsymbol{y}} \right), \\ \boldsymbol{K}^* \boldsymbol{\eta}(\boldsymbol{x}) &:= \boldsymbol{\pi}_\tau^+ \left(\boldsymbol{x} \mapsto \mathbf{curl}_{\boldsymbol{x}} \int_\Gamma E(\boldsymbol{x}, \boldsymbol{y})\, \boldsymbol{n} \times \boldsymbol{\eta}(\boldsymbol{y})\, ds_{\boldsymbol{y}} \right) - \boldsymbol{\eta}(\boldsymbol{x}), \\ \boldsymbol{W\eta}(\boldsymbol{x}) &:= \boldsymbol{\gamma}_\tau^+ \left[\boldsymbol{x} \mapsto \mathbf{curl}_{\boldsymbol{x}} \left(\mathbf{curl}_{\boldsymbol{x}} \int_\Gamma E(\boldsymbol{x}, \boldsymbol{y})\, \boldsymbol{n} \times \boldsymbol{\eta}(\boldsymbol{y})\, ds_{\boldsymbol{y}} \right) \right]. \end{aligned}$$

We present the main properties of these operators in the following theorem.

Theorem 5.4.1 *The mappings*

$$
\begin{aligned}
&S : \mathrm{H}^{-1/2}(\Gamma) \to \mathrm{H}^{1/2}(\Gamma),\\
&\boldsymbol{V} : \mathbf{H}_{\parallel}^{-1/2}(\Gamma) \to \mathbf{H}_{\parallel}^{1/2}(\Gamma),\\
&\boldsymbol{K} : \mathbf{H}^{-1/2}\left(\mathrm{div}_\Gamma;\Gamma\right) \to \mathbf{H}^{-1/2}\left(\mathrm{div}_\Gamma;\Gamma\right),\\
&\boldsymbol{K}^* : \mathbf{H}^{-1/2}\left(\mathrm{curl}_\Gamma;\Gamma\right) \to \mathbf{H}^{-1/2}\left(\mathrm{curl}_\Gamma;\Gamma\right),\\
&\boldsymbol{W} : \mathbf{H}^{-1/2}\left(\mathrm{curl}_\Gamma;\Gamma\right) \to \mathbf{H}^{-1/2}\left(\mathrm{div}_\Gamma;\Gamma\right)
\end{aligned}
$$

are well-defined, linear, bounded and satisfy the following properties:

- *There exist* $\alpha_1 > 0$ *and* $\alpha_2 > 0$ *such that:*

$$
\langle \phi, S\phi \rangle_{1/2,\Gamma} \geq \alpha_1 \|\phi\|^2_{-1/2,\Gamma} \qquad \forall \phi \in \mathrm{H}^{-1/2}(\Gamma) \tag{5.30}
$$

$$
\langle \boldsymbol{\eta}, \boldsymbol{V}\boldsymbol{\eta} \rangle_{\tau,\Gamma} \geq \alpha_2 \|\boldsymbol{\eta}\|^2_{\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma;\Gamma)} \qquad \forall \boldsymbol{\eta} \in \mathbf{H}^{-1/2}\left(\mathrm{div}_\Gamma 0;\Gamma\right). \tag{5.31}
$$

- *The operators* S *and* $\boldsymbol{W}$ *verify*

$$
\langle \boldsymbol{W}\boldsymbol{\lambda}, \boldsymbol{\eta} \rangle_{\tau,\Gamma} = -\left\langle \mathrm{curl}_\Gamma \boldsymbol{\eta}, S(\mathrm{curl}_\Gamma \boldsymbol{\lambda}) \right\rangle_{1/2,\Gamma} \qquad \forall \boldsymbol{\lambda},\ \boldsymbol{\eta} \in \mathbf{H}^{-1/2}\left(\mathrm{curl}_\Gamma;\Gamma\right). \tag{5.32}
$$

- *The operators* $\boldsymbol{K}$ *and* $\boldsymbol{K}^*$ *satisfy*

$$
\langle \boldsymbol{K}\boldsymbol{\eta}, \boldsymbol{\xi} \rangle_{\tau,\Gamma} = \langle \boldsymbol{\eta}, \boldsymbol{K}^*\boldsymbol{\xi} \rangle_{\tau,\Gamma} \quad \forall \boldsymbol{\eta} \in \mathbf{H}^{-1/2}\left(\mathrm{div}_\Gamma 0;\Gamma\right),\ \boldsymbol{\xi} \in \mathbf{H}^{-1/2}\left(\mathrm{curl}_\Gamma;\Gamma\right). \tag{5.33}
$$

Proof. The identity (5.33) is the equation (6.5) from [46]. The other results can be found in Theorems 6.1, 6.2 and 6.3 from the same reference. □

On the one hand, applying $\boldsymbol{\gamma}_\tau^+ \circ \mu_0^{-1}\,\mathbf{curl}$ to (5.29) and using (5.20) and (5.28), we have

$$
\mathbf{curl}_\Gamma\, \lambda = \mu_0^{-1} \boldsymbol{W} \boldsymbol{\pi}_\tau^+ \boldsymbol{u} - \boldsymbol{K}\, \mathbf{curl}_\Gamma\, \lambda. \tag{5.34}
$$

On the other hand, applying the trace operator $\boldsymbol{\pi}_\tau^+$ to (5.29) yields the boundary integral equation

$$
\boldsymbol{\pi}_\tau^+ \boldsymbol{u} = \boldsymbol{\pi}_\tau^+ \left(\boldsymbol{x} \mapsto \mathbf{curl}_{\boldsymbol{x}} \int_\Gamma E(\boldsymbol{x},\boldsymbol{y})\, \boldsymbol{n} \times \boldsymbol{\pi}_\tau^+ \boldsymbol{u}\, ds_{\boldsymbol{y}} \right) - \boldsymbol{V}\boldsymbol{\gamma}_\tau^+(\mathbf{curl}\, \boldsymbol{u}) - \mathbf{grad}_\Gamma\, S\gamma_{\boldsymbol{n}}^+ \boldsymbol{u}
$$

Then, taking into account (5.33), we deduce that

$$
\boldsymbol{K}^* \boldsymbol{\pi}_\tau^+ \boldsymbol{u} - \boldsymbol{V}\boldsymbol{\gamma}_\tau^+(\mathbf{curl}\, \boldsymbol{u}) - \mathbf{grad}_\Gamma\, S\gamma_{\boldsymbol{n}}^+ \boldsymbol{u} = \mathbf{0},
$$

or equivalently

$$\boldsymbol{K}^* \left(\mu_0^{-1}\boldsymbol{\pi}_\tau^+\boldsymbol{u}\right) - \boldsymbol{V}\boldsymbol{\gamma}_\tau^+(\mu_0^{-1}\,\mathbf{curl}\,\boldsymbol{u}) - \mu_0^{-1}\,\mathbf{grad}_\Gamma\, S\gamma_{\boldsymbol{n}}^+\boldsymbol{u} = \mathbf{0}.$$

Therefore, recalling that $\mu = \mu_0$ in Ω' and using (5.20) and (5.28), we obtain

$$\boldsymbol{K}^* \left(\mu_0^{-1}\boldsymbol{\pi}_\tau^+\boldsymbol{u}\right) - \boldsymbol{V}(\mathbf{curl}_\Gamma\,\lambda) - \mu_0^{-1}\,\mathbf{grad}_\Gamma\, S\gamma_{\boldsymbol{n}}^+\boldsymbol{u} = \mathbf{0}.$$

Hence, testing the previous identity with $\mathrm{curl}_\Gamma\boldsymbol{\eta}$, $\boldsymbol{\eta} \in \mathrm{H}_0^{1/2}(\Gamma)$, we obtain

$$-\left\langle \mathbf{curl}_\Gamma\,\eta, \boldsymbol{V}(\mathbf{curl}_\Gamma\,\lambda)\right\rangle_{\tau,\Gamma} + \mu_0^{-1}\left\langle \boldsymbol{K}(\mathbf{curl}_\Gamma\,\eta), \boldsymbol{\pi}_\tau\boldsymbol{u}\right\rangle_{\tau,\Gamma} = 0 \qquad \forall \eta \in \mathrm{H}_0^{1/2}(\Gamma).$$

Consequently, using the previous identity with (5.28), (5.32) and (5.34), we obtain the following symmetric FEM-BEM coupling for our problem:
Find $\boldsymbol{u} \in \mathcal{W}$, $p \in \mathrm{L}^2(0,T;M(\Omega_\mathrm{d}))$ and $\lambda \in \mathrm{L}^2(0,T;\mathrm{H}_0^{1/2}(\Gamma))$ such that

$$\begin{aligned}
&\frac{d}{dt}\left[(\boldsymbol{u}(t),\boldsymbol{v})_\sigma + b(\boldsymbol{v},p(t))\right] + \left(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v}\right)_{0,\Omega} \\
&\qquad + \mu_0^{-1}\left\langle S(\mathrm{curl}_\Gamma\boldsymbol{\pi}_\tau\boldsymbol{u}), \mathrm{curl}_\Gamma\boldsymbol{\pi}_\tau\boldsymbol{v}\right\rangle_{1/2,\Gamma} \\
&\qquad\qquad + \left\langle \boldsymbol{K}\,\mathbf{curl}_\Gamma\,\lambda(t), \boldsymbol{\pi}_\tau\boldsymbol{v}\right\rangle_{\tau,\Gamma} = (\boldsymbol{f}(t),\boldsymbol{v})_{0,\Omega} \qquad \forall \boldsymbol{v} \in \mathbf{H}(\mathbf{curl};\Omega), \\
&-\left\langle \mathbf{curl}_\Gamma\,\eta, \boldsymbol{V}(\mathbf{curl}_\Gamma\,\lambda)\right\rangle_{\tau,\Gamma} + \mu_0^{-1}\left\langle \boldsymbol{K}(\mathbf{curl}_\Gamma\,\eta), \boldsymbol{\pi}_\tau\boldsymbol{u}\right\rangle_{\tau,\Gamma} = 0 \qquad \forall \eta \in \mathrm{H}_0^{1/2}(\Gamma), \\
&b(\boldsymbol{u}(t),q) = 0 \qquad \forall q \in M(\Omega_\mathrm{d}), \\
&\boldsymbol{u}|_{\Omega_\mathrm{c}}(0) = \mathbf{0}.
\end{aligned} \tag{5.35}$$

5.5 Existence and uniqueness.

From now on, we assume that Ω_d satisfies the following topological assumption, which is necessary to prove Corollary 5.5.1 below: there exists a set $\{\omega_j,\, j = 1,\ldots,J\}$ of admissible cuts of Ω_d such that $\cup_{j=1}^J \partial\omega_j \subset \Sigma$ and any connected component of

$$\Omega_\mathrm{d}^0 := \Omega_\mathrm{d} \setminus \left(\cup_{j=1}^J \omega_j\right)$$

is simply connected. This assumption is satisfied for any geometry in practice.

To prove the existence and uniqueness of solution of Problem (5.35), we will deduce a reduced problem obtained from eliminating λ. To this end, using Corollary 5.2.1 and Theorem 5.4.1 we define the operator

$$\begin{aligned}
R :&\mathrm{H}^{-1/2}(\Gamma) \to \mathrm{H}_0^{1/2}(\Gamma) \\
&\xi \mapsto \theta
\end{aligned}$$

characterized by

$$\langle \mathbf{curl}_\Gamma \chi, \boldsymbol{V}(\mathbf{curl}_\Gamma \theta)\rangle_{\tau,\Gamma} = \langle \xi, \chi\rangle_{1/2,\Gamma} \qquad \forall \chi \in \mathrm{H}_0^{1/2}(\Gamma). \tag{5.36}$$

Then, from the second equation of (5.35) it is clear that $\lambda = \mu_0^{-1} R(\mathrm{curl}_\Gamma \boldsymbol{K}^* \boldsymbol{\pi}_\tau \boldsymbol{u})$. Therefore, it is sufficient to analyze the existence and uniqueness of solution for the following (reduced) problem:
Find $\boldsymbol{u} \in \mathcal{W}$, $p \in \mathrm{L}^2(0,T; M(\Omega_\mathrm{d}))$ such that:

$$\begin{aligned} &\frac{d}{dt}\left[(\boldsymbol{u}(t), \boldsymbol{v})_\sigma + b(\boldsymbol{v}, p(t))\right] \\ &\qquad + \left(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v}\right)_{0,\Omega} + c(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}(t), \boldsymbol{v})_{0,\Omega} \qquad \forall \boldsymbol{v} \in \mathbf{H}(\mathbf{curl}; \Omega), \\ &\qquad\qquad b(\boldsymbol{u}(t), q) = 0 \qquad \forall q \in M(\Omega_\mathrm{d}), \\ &\qquad\qquad \boldsymbol{u}|_{\Omega_\mathrm{c}}(0) = \mathbf{0}, \end{aligned} \tag{5.37}$$

where $c(\cdot,\cdot) : \mathbf{H}(\mathbf{curl}; \Omega) \times \mathbf{H}(\mathbf{curl}; \Omega) \to \mathbb{R}$ is the bounded, symmetric and nonnegative bilinear form given by

$$c(\boldsymbol{u}, \boldsymbol{v}) := \mu_0^{-1} \left\langle \left(\mathbf{curl}_\Gamma S \mathrm{curl}_\Gamma + \boldsymbol{K}\, \mathbf{curl}_\Gamma R \mathrm{curl}_\Gamma \boldsymbol{K}^*\right) \boldsymbol{\pi}_\tau \boldsymbol{u}, \boldsymbol{\pi}_\tau \boldsymbol{v}\right\rangle_{\tau,\Gamma}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbf{H}(\mathbf{curl}; \Omega)$. We introduce the space

$$V(\Omega_\mathrm{d}) := \{\boldsymbol{v} \in \mathbf{H}(\mathbf{curl}, \Omega_\mathrm{d}) : \ \boldsymbol{\gamma}_\tau \boldsymbol{v} = 0 \text{ on } \Sigma;\ b(\boldsymbol{v}, q) = 0\ \forall q \in M(\Omega_\mathrm{d})\}$$

In other words, since $\varepsilon(\boldsymbol{x}) = \varepsilon_0$ in Ω_d,

$$\begin{aligned} V(\Omega_\mathrm{d}) = \Big\{ \boldsymbol{v} \in \mathbf{H}(\mathbf{curl}, \Omega_\mathrm{d}) :\ & \mathrm{div}\, \boldsymbol{v} = 0 \text{ in } \Omega_\mathrm{d};\ \boldsymbol{\gamma}_\tau \boldsymbol{v} = 0 \text{ on } \Sigma;\ \gamma_{\boldsymbol{n}} \boldsymbol{v} = 0 \text{ on } \Gamma; \\ & \langle \gamma_{\boldsymbol{n}} \boldsymbol{v}, 1\rangle_{1/2,\Sigma_i} = 0,\ i = 1, \dots, I \Big\}. \end{aligned}$$

Notice that if $\boldsymbol{u}$ were in terms of the actual electric field $\boldsymbol{E}$ instead of $\boldsymbol{E}^*$, then $\boldsymbol{u}|_{\Omega_\mathrm{d}}$ would not belong to $V(\Omega_\mathrm{d})$, because it would not satisfy the boundary condition on Γ. This boundary condition will play a central role in the proof of Corollary 5.5.1 below. This is the reason why we have used the modified electric field $\boldsymbol{E}^*$.

Lemma 5.5.1 *There exists a real number $s > 1/2$ such that $V(\Omega_\mathrm{d})$ is continuously imbedded in $\mathrm{H}^s(\Omega_\mathrm{d})^3$.*

Proof. It is well know that the spaces $\mathbf{H}_0(\mathbf{curl}, \Omega_\mathrm{d}) \cap \mathrm{H}(\mathrm{div}; \Omega_\mathrm{d})$ and $\mathbf{H}(\mathbf{curl}, \Omega_\mathrm{d}) \cap \mathrm{H}_0(\mathrm{div}; \Omega_\mathrm{d})$ are continuously imbedded in $\mathrm{H}^s(\Omega_\mathrm{d})^3$, for some $s > 1/2$ (see [14, Proposition 3.7]). Let $\psi \in C_0^\infty(\Omega)$ such that $0 \leq \psi \leq 1$ and $\psi \equiv 1$ in $\overline{\Omega}_\mathrm{c}$. Then, for any $\boldsymbol{v} \in V(\Omega_\mathrm{d})$ we have $\boldsymbol{v} = \psi\boldsymbol{v} + (1-\psi)\boldsymbol{v}$, with

$$\psi\boldsymbol{v} \in \mathbf{H}_0(\mathbf{curl}, \Omega_\mathrm{d}) \cap \mathrm{H}(\mathrm{div}; \Omega_\mathrm{d}), \quad (1-\psi)\boldsymbol{v} \in \mathbf{H}(\mathbf{curl}, \Omega_\mathrm{d}) \cap \mathrm{H}_0(\mathrm{div}; \Omega_\mathrm{d}).$$

Consequently, $\boldsymbol{v} \in \mathrm{H}^s(\Omega_\mathrm{d})^3$ and there exists $C > 0$ (depending only on Ω_d) such that

$$\begin{aligned}\|\boldsymbol{v}\|_{\mathrm{H}^s(\Omega_\mathrm{d})^3} &\leq \|\psi\boldsymbol{v}\|_{\mathrm{H}^s(\Omega_\mathrm{d})^3} + \|(1-\psi)\boldsymbol{v}\|_{\mathrm{H}^s(\Omega_\mathrm{d})^3} \\ &\leq C\left(\|\psi\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl},\Omega_\mathrm{d})} + \|(1-\psi)\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl},\Omega_\mathrm{d})}\right) \\ &\leq C\|\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl},\Omega_\mathrm{d})}.\end{aligned}$$

□

Remark 5.5.1 *The previous lemma implies that the imbedding of $V(\Omega_\mathrm{d})$ into $\mathrm{L}^2(\Omega_\mathrm{d})^3$ is compact.*

We now recall one of the three statements provided by the well-known Petree-Tartar Lemma; see for instance [43, Chapter I, Theorem 2.1] or also [6, Lemma A.38].

Lemma 5.5.2 *Let X, Y and Z be three Banach spaces. Let $A : X \to Y$ and $T : X \to Z$ linear and bounded operators, with A injective and T compact. If there exists $\kappa > 0$ such that*

$$\kappa\|x\|_X \leq \|Ax\|_Y + \|Tx\|_Z \qquad \forall x \in X,$$

then there exists $\alpha > 0$ such that

$$\alpha\|x\|_X \leq \|Ax\|_Y \qquad \forall x \in X.$$

Corollary 5.5.1 *On the space $V(\Omega_\mathrm{d})$, the seminorm $\boldsymbol{v} \mapsto \|\mathbf{curl}\,\boldsymbol{v}\|_{0,\Omega_\mathrm{d}}$ is equivalent to the $\mathbf{H}(\mathbf{curl}, \Omega_\mathrm{d})$-norm.*

Proof. In order to apply Lemma 5.5.2 we let $X := V(\Omega_\mathrm{d})$, $Y = Z := \mathrm{L}^2(\Omega_\mathrm{d})^3$, and define the bounded linear operators $A : X \to Y$ and $T : X \to Z$ by

$$A\boldsymbol{v} := \mathbf{curl}\,\boldsymbol{v}, \quad T\boldsymbol{v} := \boldsymbol{v} \qquad \forall \boldsymbol{v} \in V(\Omega_\mathrm{d}).$$

Since T is compact (cf. Remark 5.5.1), we only need to prove that A is injective. In fact, since $\cup_{j=1}^{J}\partial\omega_j \subset \Sigma$, for any $\boldsymbol{v} \in V(\Omega_{\rm d})$ there exists a unique $\psi \in \mathbf{H}(\mathbf{curl},\Omega_{\rm d}) \cap \mathrm{H}(\mathrm{div};\Omega_{\rm d})$ (cf. [14, Subsection 3.5]) such that:

$$\begin{aligned}
\boldsymbol{v} = \mathbf{curl}\,\psi \ \text{ in } \Omega_{\rm d}, &\quad \mathrm{div}\,\psi = 0 \ \text{ in } \Omega_{\rm d}\\
\boldsymbol{\gamma}_\tau\psi = \mathbf{0} \ \text{ on } \Gamma, &\quad \frac{\partial\psi}{\partial\boldsymbol{n}} = 0 \ \text{ on } \Sigma,\\
\left\langle \frac{\partial\psi}{\partial\boldsymbol{n}}, 1 \right\rangle_{1/2,\Sigma_j} = 0, &\quad i = 1,\dots,I,\\
\left\langle \frac{\partial\psi}{\partial\boldsymbol{n}}, 1 \right\rangle_{1/2,\omega_j} = 0, &\quad i = 1,\dots,I.
\end{aligned}$$

Hence, if $\mathbf{curl}\,\boldsymbol{v} = \mathbf{0}$ we obtain

$$\int_{\Omega_{\rm d}} \boldsymbol{v}\cdot\boldsymbol{v} = \int_{\Omega_{\rm d}} \boldsymbol{v}\cdot\mathbf{curl}\,\psi = \langle \boldsymbol{\gamma}_\tau\boldsymbol{v}, \boldsymbol{\pi}_\tau\psi\rangle_{\tau,\Gamma} \ + \ \langle \boldsymbol{\gamma}_\tau\boldsymbol{v}, \boldsymbol{\pi}_\tau\psi\rangle_{\tau,\Sigma} = 0,$$

which implies $\boldsymbol{v} = \mathbf{0}$. □

The proof of the following results are similar to those in Section 3.4. For the sake of completeness, we present the complete details of the proofs.

Lemma 5.5.3 *The linear mapping*

$$\begin{aligned}
\mathcal{E}:\ \mathbf{H}(\mathbf{curl},\Omega_{\rm c}) &\ \rightarrow\ V(\Omega)\\
\boldsymbol{v}_{\rm c} &\ \mapsto\ \mathcal{E}\boldsymbol{v}_{\rm c}
\end{aligned}$$

characterized by $(\mathcal{E}\boldsymbol{v}_{\rm c})\,|_{\Omega_{\rm c}} = \boldsymbol{v}_{\rm c}$ *and*

$$\mu_0^{-1}\left(\mathbf{curl}\,\mathcal{E}\boldsymbol{v}_{\rm c}, \mathbf{curl}\,\boldsymbol{w}\right)_{0,\Omega_{\rm d}} + c(\mathcal{E}\boldsymbol{v}_{\rm c}, \boldsymbol{w}) = 0 \qquad \forall \boldsymbol{w} \in V(\Omega_{\rm d}) \tag{5.38}$$

is well defined and bounded.

Proof. It is well know that the linear mapping

$$\begin{aligned}
\mathcal{L}:\ \mathbf{H}(\mathbf{curl},\Omega_{\rm c}) &\ \rightarrow\ \{\boldsymbol{v}|_{\Omega_{\rm d}}:\ \boldsymbol{v} \in \mathbf{H}(\mathbf{curl};\Omega)\}\\
\boldsymbol{v}_{\rm c} &\ \mapsto\ \mathcal{L}\boldsymbol{v}_{\rm c} := (\boldsymbol{\gamma}_\tau^+)^{-1}(\boldsymbol{\gamma}_\tau^-\boldsymbol{v}_{\rm c}),
\end{aligned} \tag{5.39}$$

where $(\boldsymbol{\gamma}_\tau^+)^{-1}$ is a continuous right inverse of $\boldsymbol{\gamma}_\tau^+$ (cf. Theorem 5.2.1), is bounded and satisfies $\boldsymbol{\gamma}_\tau^+(\mathcal{L}\boldsymbol{v}_{\rm c}) = \boldsymbol{\gamma}_\tau^-\boldsymbol{v}_{\rm c}$ on Σ.

For $\boldsymbol{v}_{\rm c} \in \mathbf{H}(\mathbf{curl},\Omega_{\rm c})$, let $\boldsymbol{z} \in \mathcal{L}\boldsymbol{v}_{\rm c} + \mathbf{H}_\Sigma(\mathbf{curl};\Omega_{\rm d})$ and $r \in M(\Omega_{\rm d})$ such that

$$\begin{aligned}
\mu_0^{-1}\left(\mathbf{curl}\,\boldsymbol{z}, \mathbf{curl}\,\boldsymbol{w}\right)_{0,\Omega_{\rm d}} + c(\boldsymbol{z},\boldsymbol{w}) + b(\boldsymbol{w},r) = 0 &\qquad \forall \boldsymbol{w} \in \mathbf{H}_\Sigma(\mathbf{curl};\Omega_{\rm d}),\\
b(\boldsymbol{z},q) = 0 &\qquad \forall q \in M(\Omega_{\rm d}),
\end{aligned}$$

where, as usual,

$$\mathbf{H}_\Sigma(\mathbf{curl};\Omega_\mathrm{d}) := \{\boldsymbol{v} \in \mathbf{H}(\mathbf{curl},\Omega_\mathrm{d}) : \boldsymbol{\gamma}_\tau \boldsymbol{v} = \mathbf{0} \text{ on } \Sigma\}.$$

The well-posedness of this problem is guaranteed by the Babuška-Brezzi theory. Indeed, on the one hand, the fact that $\mathbf{grad}(M(\Omega_\mathrm{d})) \subset \mathbf{H}_\Sigma(\mathbf{curl};\Omega_\mathrm{d})$ implies easily the following inf-sup condition for b:

$$\sup_{\boldsymbol{z}\in\mathbf{H}_\Sigma(\mathbf{curl};\Omega_\mathrm{d})} \frac{b(\boldsymbol{z},q)}{\|\boldsymbol{z}\|_{\mathbf{H}(\mathbf{curl},\Omega_\mathrm{d})}} \geq \varepsilon_0 \frac{(\mathbf{grad}\, q, \mathbf{grad}\, q)_{0,\Omega_\mathrm{d}}}{\|\mathbf{grad}\, q\|_{\mathbf{H}(\mathbf{curl},\Omega_\mathrm{d})}} = \varepsilon_0 |q|_{1,\Omega_\mathrm{d}} \qquad \forall q \in M(\Omega_\mathrm{d}). \quad (5.40)$$

On the other hand, since $c(\cdot,\cdot)$ a nonnegative bilinear form, Corollary 5.5.1 ensures the ellipticity in the kernel property: there exists $C_1 > 0$ such that

$$\mu_0^{-1} (\mathbf{curl}\, \boldsymbol{w}, \mathbf{curl}\, \boldsymbol{w})_{0,\Omega_\mathrm{d}} + c(\boldsymbol{w},\boldsymbol{w}) \geq C_1 \|\boldsymbol{w}\|^2_{\mathbf{H}(\mathbf{curl},\Omega_\mathrm{d})} \qquad \forall \boldsymbol{w} \in V(\Omega_\mathrm{d}).$$

Finally, the stability results provided by the Babuška-Brezzi theory show that $\mathcal{E}$ is bounded. □

Lemma 5.5.4 *The inner product in $V(\Omega)$*

$$(\boldsymbol{u},\boldsymbol{v})_{V_0(\Omega)} := (\boldsymbol{u},\boldsymbol{v})_\sigma + \left(\mu^{-1}\, \mathbf{curl}\, \boldsymbol{u}, \mathbf{curl}\, \boldsymbol{v}\right)_{0,\Omega} + c(\boldsymbol{u},\boldsymbol{v}) \quad (5.41)$$

induces a norm $\|\cdot\|_{V(\Omega)}$ that is equivalent to the $\mathbf{H}(\mathbf{curl};\Omega)$ norm in $V(\Omega)$. Moreover, the following decomposition is orthogonal with respect to the inner product $(\cdot,\cdot)_{V(\Omega)}$:

$$V(\Omega) = \widetilde{V(\Omega_\mathrm{d})} \oplus \mathcal{E}(\mathbf{H}(\mathbf{curl},\Omega_\mathrm{c})), \quad (5.42)$$

where $\widetilde{V(\Omega_\mathrm{d})}$ is the subspace of $V(\Omega)$ obtained by extending by zero the functions of $V(\Omega_\mathrm{d})$ to the whole domain Ω.

Proof. For any $\boldsymbol{v} \in V(\Omega)$, let us denote $\boldsymbol{v}_\mathrm{c} := \boldsymbol{v}|_{\Omega_\mathrm{c}}$. Since $\boldsymbol{v} - \mathcal{E}\boldsymbol{v}_\mathrm{c} \in \widetilde{V(\Omega_\mathrm{d})}$, the triangle inequality and Corollary 5.5.1 ensure the existence of a constant $C_0 > 0$ such that

$$\|\boldsymbol{v}\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} \leq 2C_0^2 \|\mathbf{curl}(\boldsymbol{v} - \mathcal{E}\boldsymbol{v}_\mathrm{c})\|^2_{0,\Omega_\mathrm{d}} + 2\|\mathcal{E}\boldsymbol{v}_\mathrm{c}\|^2_{\mathbf{H}(\mathbf{curl};\Omega)}.$$

Hence, using again the triangle inequality and Lemma 5.5.3, we have

$$\|\boldsymbol{v}\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} \leq C_1 \left(\|\mathbf{curl}\, \boldsymbol{v}\|^2_{0,\Omega_\mathrm{d}} + \|\boldsymbol{v}_\mathrm{c}\|^2_{\mathbf{H}(\mathbf{curl};\Omega)}\right) = C_1 \left(\|\boldsymbol{v}\|^2_{0,\Omega_\mathrm{c}} + \|\mathbf{curl}\, \boldsymbol{v}\|^2_{0,\Omega}\right).$$

Consequently,

$$\|\boldsymbol{v}\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} \leq C_1 \max\{\sigma_0^{-1}, 1\}\|\boldsymbol{v}\|^2_{V(\Omega)}.$$

The other inequality is straightforward.

Finally, using (5.38) it is easy to check that $\mathcal{E}(\mathbf{H}(\mathbf{curl}, \Omega_c))$ is the orthogonal complement of $\widetilde{V(\Omega_d)}$ in $V(\Omega)$ with respect to the inner product $(\cdot,\cdot)_{V(\Omega)}$. □

Theorem 5.5.1 *The reduced problem* (5.37) *has a unique solution* $(\boldsymbol{u}, p)$*, with*

$$\max_{t\in[0,T]} \|\boldsymbol{u}(t)\|^2_{0,\Omega_c} + \int_0^T \|\boldsymbol{u}(t)\|^2_{\mathbf{H}(\mathbf{curl};\Omega)}\, dt \leq C \int_0^T \|\boldsymbol{f}(t)\|^2_{0,\Omega}\, dt, \tag{5.43}$$

for some constant $C > 0$*. Moreover, if we define* $\lambda = \mu_0^{-1} R(\mathrm{curl}_\Gamma \boldsymbol{K}^* \boldsymbol{\pi}_\tau \boldsymbol{u})$ *then* $(\boldsymbol{u}, \lambda, p)$ *is the only solution of Problem* (5.35)*.*

Proof. We first notice that the second equation of (5.37) means that $\boldsymbol{u} \in \mathcal{W}_0$. The decomposition (5.42) implies that the direct sum

$$\mathcal{W}_0 = \mathrm{L}^2(0, T; \widetilde{V(\Omega_d)}) \oplus \mathcal{E}(W^1(0, T; \mathbf{H}(\mathbf{curl}, \Omega_c)))$$

is orthogonal with respect to the inner product $\int_0^T (\cdot,\cdot)_{V(\Omega)}\, dt$. Hence, let $\boldsymbol{u} = \boldsymbol{u}_d + \mathcal{E}\boldsymbol{u}_c$, with $\boldsymbol{u}_d \in \mathrm{L}^2(0, T; \widetilde{V(\Omega_d)})$ and $\mathcal{E}\boldsymbol{u}_c \in \mathcal{E}(W^1(0, T; \mathbf{H}(\mathbf{curl}, \Omega_c)))$. Testing the first equation of (5.37) with $\boldsymbol{v} \in \widetilde{V(\Omega_d)}$, we find that the first component satisfies

$$\mu_0^{-1} (\mathbf{curl}\, \boldsymbol{u}_d(t), \mathbf{curl}\, \boldsymbol{v})_{0,\Omega_d} + c(\boldsymbol{u}_d(t), \boldsymbol{v}) = (\boldsymbol{f}(t), \boldsymbol{v})_{0,\Omega_d} \qquad \forall \boldsymbol{v} \in V(\Omega_d).$$

Since $c(\cdot,\cdot)$ is nonnegative, Corollary 5.5.1 and the Lax-Milgram lemma imply that this problem admits a unique solution and there exists $C_1 > 0$ such that

$$\int_0^T \|\boldsymbol{u}_d\|^2_{\mathbf{H}(\mathbf{curl},\Omega_d)}\, dt \leq C_1 \int_0^T \|\boldsymbol{f}(t)\|^2_{0,\Omega}\, dt. \tag{5.44}$$

Testing the first equation of (5.37) with $\mathcal{E}\boldsymbol{v}$ ($\boldsymbol{v} \in \mathbf{H}(\mathbf{curl}, \Omega_c)$), we determine the second component $\mathcal{E}\boldsymbol{u}_c$ of $\boldsymbol{u}$ by solving:

$$\frac{d}{dt}(\boldsymbol{u}_c(t), \boldsymbol{v})_\sigma + \left(\mu^{-1}\, \mathbf{curl}\, \mathcal{E}\boldsymbol{u}_c(t), \mathbf{curl}\, \mathcal{E}\boldsymbol{v}\right)_{0,\Omega} + c(\mathcal{E}\boldsymbol{u}_c(t), \mathcal{E}\boldsymbol{v}) = (\boldsymbol{f}(t), \mathcal{E}\boldsymbol{v})_{0,\Omega} \tag{5.45}$$

for all $\boldsymbol{v} \in \mathbf{H}(\mathbf{curl}, \Omega_c)$ and satisfying $\boldsymbol{u}_c(0) = \mathbf{0}$. It is immediate that

$$\left(\mu^{-1}\, \mathbf{curl}\, \mathcal{E}\boldsymbol{v}, \mathbf{curl}\, \mathcal{E}\boldsymbol{v}\right)_{0,\Omega} + c(\mathcal{E}\boldsymbol{v}, \mathcal{E}\boldsymbol{v}) + (\boldsymbol{v}, \boldsymbol{v})_\sigma \geq \min\{\sigma_0, \mu_1^{-1}\}\|\boldsymbol{v}\|^2_{\mathbf{H}(\mathbf{curl},\Omega_c)}$$

for any $\boldsymbol{v} \in \mathbf{H}(\mathbf{curl}, \Omega_{\mathrm{c}})$ and $t \in (0, T)$. Therefore, the well-posedness of the parabolic problem (5.45) follows immediately from a simple variant of Lions Theorem (see, for instance, [73, Remark 23.25]). In addition, there exists $C_2 > 0$ such that

$$\max_{t\in[0,T]} \|\boldsymbol{u}_{\mathrm{c}}(t)\|^2_{0,\Omega_{\mathrm{c}}} + \int_0^T \|\boldsymbol{u}_{\mathrm{c}}(t)\|^2_{\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{d}})}\, dt \le C_2 \int_0^T \|\boldsymbol{f}(t)\|^2_{0,\Omega}\, dt,$$

which, combined with (5.44) and the boundedness of $\mathcal{E}$, yields (5.43).

It remains to prove the existence and uniqueness of the Lagrange multiplier p. Given $q \in M(\Omega_{\mathrm{d}})$, we denote by $\widetilde{\mathbf{grad}}\, q \in \mathbf{H}(\mathbf{curl}; \Omega)$ the extension by zero of $\mathbf{grad}\, q$ to the whole Ω. Notice that the bilinear form b satisfies the inf-sup condition

$$\sup_{\boldsymbol{v}\in\mathbf{H}(\mathbf{curl};\Omega)} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)}} \ge \frac{b(\widetilde{\mathbf{grad}}\, q, q)}{\|\widetilde{\mathbf{grad}}\, q\|_{\mathbf{H}(\mathbf{curl};\Omega)}} = \varepsilon_0 |q|_{1,\Omega_{\mathrm{d}}} \qquad \forall q \in M(\Omega_{\mathrm{d}}). \tag{5.46}$$

Let us consider now $\mathcal{G} \in \mathcal{C}^0([0, T], \mathbf{H}(\mathbf{curl}; \Omega)')$ defined by

$$\begin{aligned}\langle \mathcal{G}(t), \boldsymbol{v}\rangle := &- (\boldsymbol{u}(t), \boldsymbol{v})_\sigma - \int_0^t \left(\mu^{-1}\, \mathbf{curl}\, \boldsymbol{u}(s), \mathbf{curl}\, \boldsymbol{v}\right)_{0,\Omega} ds \\ &- \int_0^t c(\boldsymbol{u}(s), \boldsymbol{v})\, ds + \int_0^t (\boldsymbol{f}(s), \boldsymbol{v})_{0,\Omega}\, ds\end{aligned}$$

for all $\boldsymbol{v} \in \mathbf{H}(\mathbf{curl}; \Omega)$. By integrating the first equation of (5.37) with respect to t and taking into account the definition (5.25) of $V(\Omega)$, we obtain

$$\langle \mathcal{G}(t), \boldsymbol{v}\rangle = 0 \qquad \forall \boldsymbol{v} \in V(\Omega).$$

Therefore, inf-sup condition (5.46) guarantees the existence of a unique $p(t) \in M(\Omega_{\mathrm{d}})$ such that (see [43, Lemma I.4.1])

$$b(\boldsymbol{v}, p(t)) = \langle \mathcal{G}(t), \boldsymbol{v}\rangle \qquad \forall \boldsymbol{v} \in \mathbf{H}(\mathbf{curl}; \Omega). \tag{5.47}$$

We deduce that $(\boldsymbol{u}, p)$ solves (5.37) by differentiating the last identity with respect to t in the sense of distributions.

Finally, if $(\boldsymbol{u}, p)$ is the solution of (5.37) and we define $\lambda = \mu_0^{-1} R(\mathrm{curl}_\Gamma \boldsymbol{K}^* \boldsymbol{\pi}_\tau \boldsymbol{u})$, then using the definition of R, it is easy to see that $(\boldsymbol{u}, \lambda, p)$ solves (5.35). Conversely, if $(\boldsymbol{u}, \lambda, p)$ is a solution of (5.35), it follows that $\lambda = \mu_0^{-1} R(\mathrm{curl}_\Gamma \boldsymbol{K}^* \boldsymbol{\pi}_\tau \boldsymbol{u})$. Substituting this relationship in the first equation of (5.35) we deduce that $(\boldsymbol{u}, p)$ solves (5.37), which proves uniqueness of solution for (5.35). □

Lemma 5.5.5 *The Lagrange multiplier p of Problem* (5.35) *vanishes identically.*

Proof. Let $q \in M(\Omega_{\mathrm{d}})$. From the compatibility conditions (5.12), it follows that

$$(\boldsymbol{f}, \mathbf{grad}\, q)_{0,\Omega_{\mathrm{d}}} = \langle \gamma_{\boldsymbol{n}} \boldsymbol{f}, q \rangle_{1/2,\partial\Omega_{\mathrm{d}}} = \sum_{i=1}^{I} q|_{\Sigma_i} \langle \gamma_{\boldsymbol{n}} \boldsymbol{f}, 1 \rangle_{1/2,\Sigma_i} = 0.$$

Therefore, testing the first equation of (5.35) with $\mathbf{grad}\, q$ (extended by zero to the whole Ω) we deduce

$$\frac{d}{dt} b(\mathbf{grad}\, q, p(t)) + \langle \boldsymbol{K}\, \mathbf{curl}_\Gamma\, \lambda(t), \mathbf{grad}_\Gamma\, q \rangle_{\tau,\Gamma} = (\boldsymbol{f}(t), \mathbf{grad}\, q)_{0,\Omega_{\mathrm{d}}} = 0$$

On the other hand, taking $\operatorname{div}_\Gamma$ in (5.34), we obtain $\operatorname{div}_\Gamma \boldsymbol{K}\, \mathbf{curl}_\Gamma\, \boldsymbol{\lambda} = 0$. Hence, the last identity implies $\frac{d}{dt} b(\mathbf{grad}\, q, p(t)) = 0$. Setting $t = 0$ in (5.47) and using the fact that $\mathcal{G}(0) = \mathbf{0}$, we deduce that $t \mapsto b(\mathbf{grad}\, \vartheta, \lambda(t))$ vanishes identically in $[0, T]$ for all $\vartheta \in M(\Omega_{\mathrm{d}})$. In particular $\varepsilon_0 |\lambda(t)|^2_{1,\Omega_{\mathrm{d}}} = b(\mathbf{grad}\, \lambda(t), \lambda(t)) = 0$ for all $t \in [0, T]$, and the result follows. □

Theorem 5.5.2 *If* $(\boldsymbol{u}, \lambda, p)$ *is the solution of Problem* (5.35)*, then*

$$\boldsymbol{\gamma}_\tau \left(\mu_0^{-1}\, \mathbf{curl}\, \boldsymbol{u} \right) = \mathbf{curl}_\Gamma\, \lambda \qquad \textit{in } \mathbf{H}^{-1/2}(\operatorname{div}_\Gamma; \Gamma)\,. \tag{5.48}$$

Proof. Testing the first equation of (5.35) with $\boldsymbol{v} \in C_0^\infty(\Omega_{\mathrm{d}})$ and using the previous lemma, we obtain

$$\mathbf{curl}(\mu^{-1}\, \mathbf{curl}\, \boldsymbol{u})|_{\Omega_{\mathrm{d}}} = \boldsymbol{f}|_{\Omega_{\mathrm{d}}}.$$

Testing the first equation of (5.35) with $\boldsymbol{v} \in C^\infty(\overline{\Omega}_{\mathrm{d}}) \cap \mathbf{H}_\Sigma(\mathbf{curl}; \Omega_{\mathrm{d}})$ and using the previous equality and (5.32), we deduce

$$\boldsymbol{\gamma}_\tau(\mu_0^{-1}\, \mathbf{curl}\, \boldsymbol{u}) = \mu_0^{-1} \boldsymbol{W} \boldsymbol{\pi}_\tau \boldsymbol{u} - \boldsymbol{K}\, \mathbf{curl}_\Gamma\, \lambda \qquad \text{in } \mathbf{H}^{-1/2}(\operatorname{div}_\Gamma; \Gamma)\,. \tag{5.49}$$

Hence, testing the first equation of (5.35) with $\boldsymbol{v} = \widetilde{\mathbf{grad}}\, q$, $q \in M(\Omega_{\mathrm{d}})$, and recalling (5.27) and (5.32) again, we have

$$\operatorname{div}_\Gamma(\boldsymbol{\gamma}_\tau(\mu_0^{-1}\, \mathbf{curl}\, \boldsymbol{u})) = 0. \tag{5.50}$$

The second equation of (5.35) implies

$$\boldsymbol{V}(\mathbf{curl}_\Gamma\, \lambda) - \mu_0^{-1} \boldsymbol{K}^* \boldsymbol{\pi}_\tau \boldsymbol{u} \in \mathbf{H}^{-1/2}(\operatorname{curl}_\Gamma; \Gamma) \cap \ker(\operatorname{curl}_\Gamma).$$

Then, there exists $\varphi \in \mathrm{H}^{1/2}(\Gamma)$ such that (cf. Theorem 5.1 of [34])

$$\boldsymbol{V}(\mathbf{curl}_\Gamma\, \lambda) - \mu_0^{-1} \boldsymbol{K}^* \boldsymbol{\pi}_\tau \boldsymbol{u} = \mathbf{grad}_\Gamma\, \varphi.$$

From this equation and the definition of $\boldsymbol{K}^*$, we obtain

$$\boldsymbol{\pi}_\tau \boldsymbol{u} = \boldsymbol{\pi}_\tau \left(\boldsymbol{x} \mapsto \mathbf{curl}_{\boldsymbol{x}} \int_\Gamma E(\boldsymbol{x}, \boldsymbol{y})\, \boldsymbol{n} \times \boldsymbol{\pi}_\tau \boldsymbol{u}(\boldsymbol{y})\, ds_{\boldsymbol{y}} \right) - \mu_0 \boldsymbol{V}(\mathbf{curl}_\Gamma \lambda) + \mu_0\, \mathbf{grad}_\Gamma \varphi. \quad (5.51)$$

Let $\psi \in W^1(\Omega')$ such that

$$\begin{aligned} -\Delta\psi &= 0 && \text{in } \Omega', \\ \psi &= \varphi && \text{on } \Gamma, \\ \psi(\boldsymbol{x}) &= O\left(\frac{1}{|\boldsymbol{x}|}\right) && \text{as } |\boldsymbol{x}| \to \infty, \end{aligned}$$

and consider $\boldsymbol{z} : \Omega' \to \mathbb{R}^3$ defined by

$$\boldsymbol{z}(\boldsymbol{x}) := \mathbf{curl}_{\boldsymbol{x}} \int_\Gamma E(\boldsymbol{x}, \boldsymbol{y})\, \boldsymbol{n} \times \boldsymbol{\pi}_\tau \boldsymbol{u}(\boldsymbol{y})\, ds_{\boldsymbol{y}} - \mu_0 \int_\Gamma E(\boldsymbol{x}, \boldsymbol{y})\, \mathbf{curl}_\Gamma \lambda(\boldsymbol{y})\, ds_{\boldsymbol{y}} + \mu_0\, \mathbf{grad}_\Gamma \varphi. \quad (5.52)$$

Therefore, it follows that

$$\boldsymbol{\pi}_\tau \boldsymbol{z} = \boldsymbol{\pi}_\tau \boldsymbol{u} \quad \text{and} \quad \mu_0^{-1} \boldsymbol{\gamma}_\tau\, \mathbf{curl}\, \boldsymbol{z} = \mu_0^{-1} \boldsymbol{\gamma}_\tau\, \mathbf{curl}\, \boldsymbol{u}. \quad (5.53)$$

from (5.51) and (5.49) respectively. Moreover, since

$$\mathrm{div}_{\boldsymbol{x}} \int_\Gamma E(\boldsymbol{x}, \boldsymbol{y})\, \mathbf{curl}_\Gamma \lambda(\boldsymbol{y})\, ds_{\boldsymbol{y}} = \int_\Gamma E(\boldsymbol{x}, \boldsymbol{y}) \mathrm{div}_\Gamma(\mathbf{curl}_\Gamma \lambda(\boldsymbol{y}))\, ds_{\boldsymbol{y}} = 0,$$

we deduce $\mathrm{div}\, \boldsymbol{z} = 0$ in Ω'. Hence, from (5.52), we deduce $\mathbf{curl}\,\mathbf{curl}\, \boldsymbol{z} = \mathbf{0}$ in Ω' and $\boldsymbol{z}(\boldsymbol{x}) = O\left(\frac{1}{|\boldsymbol{x}|}\right)$, $\mathbf{curl}\, \boldsymbol{z}(\boldsymbol{x}) = O\left(\frac{1}{|\boldsymbol{x}|}\right)$ as $|\boldsymbol{x}| \to \infty$. Consequently, $\boldsymbol{z}$ has the following integral representation in Ω':

$$\begin{aligned} \boldsymbol{z}(\boldsymbol{x}) := {} & \mathbf{curl}_{\boldsymbol{x}} \int_\Gamma E(\boldsymbol{x}, \boldsymbol{y})\, \boldsymbol{n} \times \boldsymbol{\pi}_\tau \boldsymbol{u}(\boldsymbol{y})\, ds_{\boldsymbol{y}} - \int_\Gamma E(\boldsymbol{x}, \boldsymbol{y}) \boldsymbol{\gamma}_\tau(\mathbf{curl}\, \boldsymbol{u}(\boldsymbol{y}))\, ds_{\boldsymbol{y}} \\ & + \mathbf{grad}_{\boldsymbol{x}} \int_\Gamma E(\boldsymbol{x}, \boldsymbol{y}) \gamma_{\boldsymbol{n}} \boldsymbol{z}. \end{aligned}$$

Applying $\boldsymbol{\pi}_\tau$ to the previous identity, we deduce

$$\boldsymbol{\pi}_\tau \boldsymbol{z} = \boldsymbol{\pi}_\tau \left(\boldsymbol{x} \mapsto \mathbf{curl}_{\boldsymbol{x}} \int_\Gamma E(\boldsymbol{x}, \boldsymbol{y})\, \boldsymbol{n} \times \boldsymbol{\pi}_\tau \boldsymbol{u}(\boldsymbol{y})\, ds_{\boldsymbol{y}} \right) - \boldsymbol{V} \boldsymbol{\gamma}_\tau(\mathbf{curl}\, \boldsymbol{u}) + \mathbf{grad}_\Gamma S(\gamma_{\boldsymbol{n}} \boldsymbol{z}).$$

Hence, using (5.51) and (5.53), we have

$$\boldsymbol{V}(\mu_0\, \mathbf{curl}_\Gamma \lambda - \boldsymbol{\gamma}_\tau(\mathbf{curl}\, \boldsymbol{u})) = \mathbf{grad}_\Gamma(\mu_0 \varphi - S(\gamma_{\boldsymbol{n}} \boldsymbol{z})).$$

Testing the previous equality with $\mu_0 \operatorname{\mathbf{curl}}_\Gamma \lambda - \boldsymbol{\gamma}_\tau(\operatorname{\mathbf{curl}} \boldsymbol{u}) \in \mathbf{H}^{-1/2}(\operatorname{div}_\Gamma 0; \Gamma)$ (cf. (5.50)) and using (5.30), we obtain

$$\begin{aligned}
&\alpha_2 \|\mu_0 \operatorname{\mathbf{curl}}_\Gamma \lambda - \boldsymbol{\gamma}_\tau(\operatorname{\mathbf{curl}} \boldsymbol{u})\|^2_{\mathbf{H}^{-1/2}(\operatorname{div}_\Gamma;\Gamma)} \\
&\qquad \le \langle \mu_0 \operatorname{\mathbf{curl}}_\Gamma \lambda - \boldsymbol{\gamma}_\tau(\operatorname{\mathbf{curl}} \boldsymbol{u}), \boldsymbol{V}(\mu_0 \operatorname{\mathbf{curl}}_\Gamma \lambda - \boldsymbol{\gamma}_\tau(\operatorname{\mathbf{curl}} \boldsymbol{u}))\rangle_{\tau,\Gamma} \\
&\qquad = \langle \mu_0 \operatorname{\mathbf{curl}}_\Gamma \lambda - \boldsymbol{\gamma}_\tau(\operatorname{\mathbf{curl}} \boldsymbol{u}), \operatorname{\mathbf{grad}}_\Gamma(\mu_0 \varphi - S\gamma_{\boldsymbol{n}} \boldsymbol{z})\rangle_{\tau,\Gamma} \\
&\qquad = 0,
\end{aligned}$$

which implies (5.48). $\square$

5.6 Analysis of the semi-discrete scheme

Let $\{\mathcal{T}_h\}_h$ be a regular family of tetrahedral meshes of Ω such that each element $K \in \mathcal{T}_h$ is contained either in $\overline{\Omega}_\mathrm{c}$ or in $\overline{\Omega}_\mathrm{d}$. As usual, h stands for the largest diameter of the tetrahedra K in $\mathcal{T}_h$. Furthermore, we suppose that the family of triangulations $\{\mathcal{T}_h(\Sigma)\}_h$ induced by $\{\mathcal{T}_h\}_h$ on Σ is quasi-uniform. From now on C denotes a positive constant, independent of h and the functions involved, which may take different values at different occurrences.

We define a semi-discrete version of (5.35) by means of Nédélec finite elements. The local representation of the mth-order element of this family on a tetrahedron K is given by (see [57, Section 5.5])

$$\mathcal{N}_m(K) := \mathbb{P}^3_{m-1} \oplus S_m,$$

where $\mathbb{P}_m$ is the set of polynomials of degree not greater than m and

$$S_m := \left\{ p \in \widetilde{\mathbb{P}}^3_m : \ \boldsymbol{x} \cdot p(\boldsymbol{x}) = 0 \right\},$$

with $\widetilde{\mathbb{P}}_m$ being the set of homogeneous polynomials of degree m. The degrees of freedom of $\mathcal{N}_m(K)$ are given by

$$\mathcal{M}_1(\boldsymbol{v}) := \left\{ \int_E \boldsymbol{v} \cdot \boldsymbol{t}_e \, q \text{ for all } q \in \mathbb{P}_{m-1} \text{ for the six edges } E \text{ of } K \right\}, \tag{5.54}$$

where $\boldsymbol{t}_E$ is a unit tangent vector along E; when $m \ge 2$ one has to add

$$\mathcal{M}_2(\boldsymbol{v}) := \left\{ \int_F (\boldsymbol{v} \times \boldsymbol{n}) \cdot \boldsymbol{q} \text{ for all } \boldsymbol{q} \in \mathbb{P}^2_{m-2} \text{ for the four faces } F \text{ of } K \right\}; \tag{5.55}$$

and finally for $m \geq 3$ one has to take also

$$\mathcal{M}_3(\boldsymbol{v}) := \left\{ \int_K \boldsymbol{v} \cdot \boldsymbol{q} \text{ for all } \boldsymbol{q} \in \mathbb{P}^3_{m-3} \right\}. \tag{5.56}$$

Nédélec [61] has proved that these degrees of freedom are "curl-conforming" and determine a unique element of $\mathcal{N}_m(K)$. Then, for any smooth enough function $\boldsymbol{v}$ on K such that the moments (5.54)-(5.56) are well defined, we can define $\mathcal{I}_K \boldsymbol{v} \in X_h(\Omega)$ characterized by

$$\mathcal{M}_i(\boldsymbol{v}) = \mathcal{M}_i(\mathcal{I}_K \boldsymbol{v}) \qquad i = 1, 2, 3.$$

The corresponding global space $X_h(\Omega)$ to approximate $\mathbf{H}(\mathbf{curl}; \Omega)$ is the space of functions that are locally in $\mathcal{N}_m(K)$ and have continuous tangential components across the faces of the triangulation $\mathcal{T}_h$:

$$X_h(\Omega) := \{ \boldsymbol{v} \in \mathbf{H}(\mathbf{curl}; \Omega) : \ \boldsymbol{v}|_K \in \mathcal{N}_m(K) \ \forall K \in \mathcal{T} \}.$$

Moreover, for any smooth enough function $\boldsymbol{v}$ on Ω, we can define a global interpolation operator $\mathcal{I}_h \boldsymbol{v} \in X_h(\Omega)$ by

$$(\mathcal{I}_h \boldsymbol{v})|_K = \mathcal{I}_K \boldsymbol{v} \qquad \forall K \in \mathcal{T}_h.$$

On the other hand, we use standard mth-order Lagrange finite elements to approximate $M(\Omega_{\mathrm{d}})$ and $\mathrm{H}^{1/2}_0(\Gamma)$:

$$M_h(\Omega_{\mathrm{d}}) := \left\{ q \in \mathrm{H}^1(\Omega_{\mathrm{d}}) : \ q|_K \in \mathbb{P}_m \ \forall K \in \mathcal{T}_h, \ \int_{\Omega_{\mathrm{d}}} q = 0, \ q|_{\Sigma_i} = C_i, \ i = 1, \dots, I \right\}$$

and

$$\Lambda_h(\Gamma) := \left\{ \vartheta \in \mathrm{H}^{1/2}_0(\Gamma) : \ \vartheta|_F \in \mathbb{P}_m \ \forall F \in \mathcal{T}_h(\Gamma) \right\}$$

We introduce the following semi-discretization of problem (5.35):
Find $\boldsymbol{u}_h(t) : [0, T] \to X_h(\Omega)$, $\lambda_h(t) : [0, T] \to \Lambda_h(\Gamma)$ and $p_h(t) : [0, T] \to M_h(\Omega_{\mathrm{d}})$ such that:

$$\begin{aligned}
\frac{d}{dt} [(\boldsymbol{u}_h(t), \boldsymbol{v})_\sigma + b(\boldsymbol{v}, p_h(t))] + \left(\mu^{-1} \, \mathbf{curl} \, \boldsymbol{u}_h, \mathbf{curl} \, \boldsymbol{v}\right)_{0,\Omega} & \\
+ \mu_0^{-1} \left\langle S(\mathrm{curl}_\Gamma \boldsymbol{\pi}_\tau \boldsymbol{u}), \mathrm{curl}_\Gamma \boldsymbol{\pi}_\tau \boldsymbol{v} \right\rangle_{1/2,\Gamma} & \\
+ \left\langle \boldsymbol{K} \, \mathbf{curl}_\Gamma \, \lambda_h(t), \boldsymbol{\pi}_\tau \boldsymbol{v} \right\rangle_{\tau,\Gamma} = (\boldsymbol{f}(t), \boldsymbol{v})_{0,\Omega} & \qquad \forall \boldsymbol{v} \in X_h(\Omega), \\
- \left\langle \mathbf{curl}_\Gamma \, \eta, \boldsymbol{V}(\mathbf{curl}_\Gamma \, \lambda_h) \right\rangle_{\tau,\Gamma} + \mu_0^{-1} \left\langle \boldsymbol{K}(\mathbf{curl}_\Gamma \, \eta), \boldsymbol{\pi}_\tau \boldsymbol{u}_h \right\rangle_{\tau,\Gamma} = 0 & \qquad \forall \eta \in \Lambda_h(\Gamma), \\
b(\boldsymbol{u}_h(t), q) = 0 & \qquad \forall q \in M_h(\Omega_{\mathrm{d}}), \\
\boldsymbol{u}_h|_{\Omega_{\mathrm{c}}}(0) = \mathbf{0}. &
\end{aligned} \tag{5.57}$$

Remark 5.6.1 *For piecewise smooth functions, the boundary integral operators in* (5.57) *are structurally equal to those for second order elliptic problems (see, for instance, [64]). The terms involving the operator S and $\boldsymbol{V}$ are immediately written in terms of integrals. The same happens with the terms involving $\boldsymbol{K}$. In fact, for any $\eta \in \Lambda_h(\Gamma)$ and $\boldsymbol{v} \in X_h(\Omega)$, we have ([46])*

$$\begin{aligned}\langle \boldsymbol{K}\,\mathbf{curl}_\Gamma\,\eta, \boldsymbol{\pi}_\tau \boldsymbol{v}\rangle_{\tau,\Gamma} = &\int_\Gamma \int_\Gamma \mathbf{curl}_\Gamma\,\eta(\boldsymbol{y}) \cdot \boldsymbol{\pi}_\tau \boldsymbol{v}(\boldsymbol{x})\, \frac{\partial E(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}(\boldsymbol{x})}\, ds_{\boldsymbol{y}}\, ds_{\boldsymbol{x}} \\ &+ \int_\Gamma \int_\Gamma \mathbf{grad}_{\boldsymbol{x}}\, E(\boldsymbol{x},\boldsymbol{y})(\mathbf{curl}_\Gamma\,\eta(\boldsymbol{y}) \cdot \boldsymbol{n}(\boldsymbol{x})) \cdot \boldsymbol{\pi}_\tau \boldsymbol{v}(\boldsymbol{x})\, ds_{\boldsymbol{y}}\, ds_{\boldsymbol{x}} \\ &- \frac{1}{2} \int_\Gamma \mathbf{curl}_\Gamma\,\eta(\boldsymbol{x}) \cdot \boldsymbol{\pi}_\tau \boldsymbol{v}(\boldsymbol{x})\, ds_{\boldsymbol{x}}.\end{aligned}$$

In order to prove the existence and uniqueness of solution of (5.57), we proceed as in the continuous case. In fact, let

$$\begin{aligned} R_h :& \mathrm{H}^{-1/2}(\Gamma) \to \Lambda_h(\Gamma) \\ & \xi \mapsto \theta \end{aligned}$$

characterized by

$$\langle \mathbf{curl}_\Gamma\,\chi, \boldsymbol{V}(\mathbf{curl}_\Gamma\,\theta)\rangle_{\tau,\Gamma} = \langle \xi, \chi\rangle_{1/2,\Gamma} \qquad \forall \chi \in \Lambda_h(\Gamma). \tag{5.58}$$

Let us remark that (5.58) is the Galerkin discrete scheme of (5.36). Consequently, using Corollary 5.2.1, we obtain the following Céa's estimate

$$\|R\xi - R_h\xi\|_{1/2,\Gamma} \le C \inf_{\eta \in \Lambda_h(\Gamma)} \|R\xi - \eta\|_{1/2,\Gamma} \qquad \forall \xi \in \mathrm{H}^{-1/2}(\Gamma). \tag{5.59}$$

Furthermore, from the second equation of (5.57) we have that

$$\lambda_h = \mu_0^{-1} R_h(\mathrm{curl}_\Gamma \boldsymbol{K}^* \boldsymbol{\pi}_\tau \boldsymbol{u}_h).$$

Hence, we only need to study the problem:
Find $\boldsymbol{u}_h : [0,T] \to X_h(\Omega)$, $p_h : [0,T] \to M_h(\Omega_\mathrm{d})$ such that:

$$\begin{aligned} &\frac{d}{dt}\left[(\boldsymbol{u}_h(t), \boldsymbol{v})_\sigma + b(\boldsymbol{v}, p_h(t))\right] \\ &\quad + \left(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}_h, \mathbf{curl}\,\boldsymbol{v}\right)_{0,\Omega} + c_h(\boldsymbol{u}_h, \boldsymbol{v}) = (\boldsymbol{f}(t), \boldsymbol{v})_{0,\Omega} \qquad \forall \boldsymbol{v} \in X_h(\Omega), \\ &\qquad b(\boldsymbol{u}_h(t), q) = 0 \qquad \forall q \in M_h(\Omega_\mathrm{d}), \\ &\qquad \boldsymbol{u}_h|_{\Omega_\mathrm{c}}(0) = \boldsymbol{0}, \end{aligned} \tag{5.60}$$

where $c_h(\cdot,\cdot) : X_h(\Omega) \times X_h(\Omega) \to \mathbb{R}$ is the uniformly bounded and nonnegative bilinear form given by:

$$c_h(\boldsymbol{u},\boldsymbol{v}) := \mu_0^{-1} \left\langle (\mathbf{curl}_\Gamma S \mathrm{curl}_\Gamma + \boldsymbol{K}\,\mathbf{curl}_\Gamma R_h \mathrm{curl}_\Gamma \boldsymbol{K}^*)\,\boldsymbol{\pi}_\tau \boldsymbol{u}, \boldsymbol{\pi}_\tau \boldsymbol{v} \right\rangle_{\tau,\Gamma} \qquad \forall \boldsymbol{u},\boldsymbol{v} \in X_h(\Omega).$$

Notice that the discrete kernel

$$V_h(\Omega) := \{\boldsymbol{v} \in X_h(\Omega) : \ b(\boldsymbol{v},q) = 0 \ \forall q \in M_h(\Omega_\mathrm{d})\}$$

is not necessarily a subspace of $V(\Omega)$. We introduce

$$V_h(\Omega_\mathrm{d}) := \{\boldsymbol{v}|_{\Omega_\mathrm{d}} : \ \boldsymbol{v} \in V_h(\Omega)\} \cap \mathbf{H}_\Sigma(\mathbf{curl};\Omega_\mathrm{d}).$$

The following is a simple variation of Proposition 4.6 from [14].

Proposition 5.6.1 *On the space $V_h(\Omega_\mathrm{d})$, the seminorm $\boldsymbol{w} \mapsto \|\mathbf{curl}\,\boldsymbol{w}\|_{0,\Omega_\mathrm{d}}$ is equivalent to the $\mathbf{H}(\mathbf{curl},\Omega_\mathrm{d})$-norm.*

Proof. Let $\boldsymbol{\varphi}_h \in V_h(\Omega_\mathrm{d})$. We consider $p \in M(\Omega_\mathrm{d})$ such that

$$\int_{\Omega_\mathrm{d}} \mathbf{grad}\,p \cdot \mathbf{grad}\,q = \int_{\Omega_\mathrm{d}} \boldsymbol{\varphi}_h \cdot \mathbf{grad}\,q \qquad \forall q \in M(\Omega_\mathrm{d}).$$

Then, the function $\boldsymbol{v} := \boldsymbol{\varphi}_h - \mathbf{grad}\,p \in V(\Omega_\mathrm{d})$ and satisfies:

$$\begin{aligned} \mathbf{curl}\,\boldsymbol{v} &= \mathbf{curl}\,\boldsymbol{\varphi}_h && \text{in } \Omega_\mathrm{d} \\ \mathrm{div}\,\boldsymbol{v} &= 0 && \text{in } \Omega_\mathrm{d} \\ \gamma_{\boldsymbol{n}} \boldsymbol{v} &= 0 && \text{on } \Gamma \\ \langle \gamma_{\boldsymbol{n}} \boldsymbol{v}, 1 \rangle_{1/2,\Sigma_i} &= 0 && i = 1,\dots,I, \\ \boldsymbol{\gamma}_\tau \boldsymbol{v} &= \mathbf{0} && \text{on } \Sigma. \end{aligned}$$

Hence, from Lemma 5.5.1 there exists $\delta > 0$ such that $\boldsymbol{v} \in \mathrm{H}^{1/2+\delta}(\Omega_\mathrm{d})^3$. Therefore, since $\mathbf{curl}\,\boldsymbol{v} = \mathbf{curl}\,\boldsymbol{\varphi}_h$, we can apply the interpolation operator $\mathcal{I}_h$ to $\boldsymbol{v}$ (cf. [14, Lemma 4.7]), which implies that $\mathcal{I}_h(\mathbf{grad}\,p)$ is well defined. Then, using [43, Chapter III, Proposition 5.10] we deduce that there exists a continuous piecewise polynomial p_h such that $\mathcal{I}_h(\mathbf{grad}\,p) = \mathbf{grad}\,p_h$. We claim that $p_h \in M_h(\Omega_\mathrm{d})$. In fact, since $\boldsymbol{v} \in \mathbf{H}_\Sigma(\mathbf{curl};\Omega_\mathrm{d})$ and $\boldsymbol{\varphi}_h \in X_{h,\Sigma}(\Omega_\mathrm{d}) := X_h(\Omega_\mathrm{d}) \cap \mathbf{H}_\Sigma(\mathbf{curl};\Omega_\mathrm{d})$, we have $\mathbf{grad}\,p_h = \mathcal{I}_h(\mathbf{grad}\,p) = \boldsymbol{\varphi}_h - \mathcal{I}_h \boldsymbol{v} \in X_{h,\Sigma}(\Omega_\mathrm{d})$ (cf. [57, Lemma 5.44]). Consequently, $\mathbf{grad}_\Gamma\,p_h = 0$ on Σ, which implies $p_h \in M_h(\Omega_\mathrm{d})$.

Thus, since $\boldsymbol{\varphi}_h \in V_h(\Omega_{\rm d})$, we obtain

$$\int_{\Omega_{\rm d}} |\boldsymbol{\varphi}_h|^2 = \int_{\Omega_{\rm d}} \boldsymbol{\varphi}_h \cdot \mathcal{I}_h \boldsymbol{\varphi}_h = \int_{\Omega_{\rm d}} \boldsymbol{\varphi}_h \cdot (\mathbf{grad}\, p_h + \mathcal{I}_h \boldsymbol{v}) = \int_{\Omega_{\rm d}} \boldsymbol{\varphi}_h \cdot \mathcal{I}_h \boldsymbol{v},$$

so that

$$\|\boldsymbol{\varphi}_h\|_{0,\Omega_{\rm d}} \leq \|\mathcal{I}_h \boldsymbol{v}\|_{0,\Omega_{\rm d}}. \tag{5.61}$$

In order to bound the right-hand side of the previous inequality, we notice that proceeding as in [14, Proposition 4.6] we get the estimate

$$\|\mathcal{I}_h \boldsymbol{v}\|_{0,\Omega_{\rm d}} \leq C \left(h \|\mathbf{curl}\, \boldsymbol{\varphi}_h\|_{0,\Omega_{\rm d}} + \|\boldsymbol{v}\|_{\mathrm{H}^{1/2+\delta}(\Omega_{\rm d})^3} \right).$$

Then, using Lemma 5.5.1 and Corollary 5.5.1, and recalling that $\mathbf{curl}\, \boldsymbol{v} = \mathbf{curl}\, \boldsymbol{\varphi}_h$, we deduce

$$\|\mathcal{I}_h \boldsymbol{v}\|_{0,\Omega_{\rm d}} \leq C(1+h) \|\mathbf{curl}\, \boldsymbol{\varphi}_h\|_{0,\Omega_{\rm d}}.$$

Finally, combining this inequality with (5.61), we deduce

$$\|\boldsymbol{\varphi}_h\|_{0,\Omega_{\rm d}} \leq C(1+h) \|\mathbf{curl}\, \boldsymbol{\varphi}_h\|_{0,\Omega_{\rm d}}.$$

□

Since $\mathbf{grad}(M_h(\Omega_{\rm d})) \subset X_{h,\Sigma}(\Omega_{\rm d})$, we have that b satisfies the discrete inf-sup condition

$$\sup_{\boldsymbol{z} \in \mathbf{H}_\Sigma(\mathbf{curl};\Omega_{\rm d})} \frac{b(\boldsymbol{z}, q)}{\|\boldsymbol{z}\|_{\mathbf{H}(\mathbf{curl},\Omega_{\rm d})}} \geq \varepsilon_0 \frac{(\mathbf{grad}\, q, \mathbf{grad}\, q)_{0,\Omega_{\rm d}}}{\|\,\mathbf{grad}\, q\|_{\mathbf{H}(\mathbf{curl},\Omega_{\rm d})}} = \varepsilon_0 |q|_{1,\Omega_{\rm d}} \qquad \forall \vartheta \in M(\Omega_{\rm d}). \tag{5.62}$$

Then, we obtain the discrete version of Lemma 5.5.3 and Lemma 5.5.4, and consequently we proceed as in the continuous case to prove the well-possedness of Problem 5.57. We do not include the proofs of the following results, since they are similar to those of Lemma 5.3, Lemma 5.4 and Theorem 5.5 from [1].

Lemma 5.6.1 *The linear mapping*

$$\begin{aligned} \mathcal{E}_h : X_h(\Omega_{\rm c}) &\to V_h(\Omega) \\ \boldsymbol{v}_{\rm c} &\mapsto \mathcal{E}_h \boldsymbol{v}_{\rm c} \end{aligned}$$

characterized by $(\mathcal{E}_h \boldsymbol{v}_{\rm c})|_{\Omega_{\rm c}} = \boldsymbol{v}_{\rm c}$ *and*

$$\mu_0^{-1} (\mathbf{curl}\, \mathcal{E}_h \boldsymbol{v}_{\rm c}, \mathbf{curl}\, \boldsymbol{w})_{0,\Omega_{\rm d}} + c_h(\mathcal{E}_h \boldsymbol{v}_{\rm c}, \boldsymbol{w}) = 0 \qquad \forall \boldsymbol{w} \in V_h(\Omega_{\rm d}) \tag{5.63}$$

is well defined and bounded uniformly in h*. Furthermore, the inner product in* $V_h(\Omega)$

$$(\boldsymbol{u},\boldsymbol{v})_{V_h(\Omega)} := (\boldsymbol{u},\boldsymbol{v})_\sigma + \left(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u},\mathbf{curl}\,\boldsymbol{v}\right)_{0,\Omega_{\mathrm{d}}} + c_h(\boldsymbol{u},\boldsymbol{v}) \tag{5.64}$$

induces a norm $\|\cdot\|_{V_h(\Omega)}$ *that is equivalent to the* $\mathbf{H}(\mathbf{curl};\Omega)$ *norm in* $V_h(\Omega)$*. Moreover, the following decomposition is orthogonal with respect to the inner product* $(\cdot,\cdot)_{V_h(\Omega)}$*:*

$$V(\Omega) = \widetilde{V_h(\Omega_{\mathrm{d}})} \oplus \mathcal{E}_h(\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}})),$$

where $\widetilde{V_h(\Omega_{\mathrm{d}})}$ *is the subspace of* $V_h(\Omega)$ *obtained by extending by zero the functions of* $V_h(\Omega_{\mathrm{d}})$ *to the whole domain* Ω*.*

Theorem 5.6.1 *Problem 5.60 has a unique solution* $(\boldsymbol{u}_h,p_h)$*. Moreover, if we define* $\lambda_h := \mu_0^{-1}R_h(\mathrm{curl}_\Gamma\boldsymbol{K}^*\boldsymbol{\pi}_\tau\boldsymbol{u}_h)$ *then* $(\boldsymbol{u}_h,\lambda_h,p_h)$ *is the only solution of Problem* (5.57)*.*

5.6.1 Error estimates.

Consider the linear projection operator $\Pi_h:\mathbf{H}(\mathbf{curl};\Omega)\to V_h(\Omega)$ defined by

$$\Pi_h\boldsymbol{v}\in V_h(\Omega): \qquad (\Pi_h\boldsymbol{v},\boldsymbol{z})_{\mathbf{H}(\mathbf{curl};\Omega)} = (\boldsymbol{v},\boldsymbol{z})_{\mathbf{H}(\mathbf{curl};\Omega)} \qquad \forall\boldsymbol{z}\in V_h(\Omega). \tag{5.65}$$

Proceeding as in Lemma 3.5.5, we deduce the following projection error estimate

$$\|\boldsymbol{v}-\Pi_h\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)} \le \inf_{\boldsymbol{z}\in X_h(\Omega)}\|\boldsymbol{v}-\boldsymbol{z}\|_{\mathbf{H}(\mathbf{curl};\Omega)} \qquad \forall\boldsymbol{v}\in V(\Omega). \tag{5.66}$$

We denote

$$\begin{aligned}
&a(\boldsymbol{v},\boldsymbol{w}) := \left(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{v},\mathbf{curl}\,\boldsymbol{w}\right)_{0,\Omega},\\
&\boldsymbol{\rho}_h(t) := \boldsymbol{u}(t)-\Pi_h\boldsymbol{u}(t),\\
&\boldsymbol{\delta}_h(t) := \Pi_h\boldsymbol{u}(t)-\boldsymbol{u}_h(t)
\end{aligned}$$

and

$$\beta_h(\boldsymbol{w}) := \|(R-R_h)\mathrm{curl}_\Gamma\boldsymbol{K}^*\boldsymbol{\pi}_\tau\boldsymbol{w}\|_{1/2,\Gamma}. \tag{5.67}$$

Notice that $\boldsymbol{\delta}_h(t)\in V_h(\Omega)$ and satisfies

$$\|\boldsymbol{\delta}_h(t)\|_{\mathbf{H}(\mathbf{curl};\Omega)} \le C\left(\|\boldsymbol{\delta}_h(t)\|_{0,\Omega_{\mathrm{c}}} + \|\mathbf{curl}\,\boldsymbol{\delta}_h(t)\|_{0,\Omega}\right) \qquad \forall t\in[0,T]. \tag{5.68}$$

In fact, let $\boldsymbol{v}\in V_h(\Omega)$. Since $\boldsymbol{v}-\mathcal{E}_h(\boldsymbol{v}|_{\Omega_{\mathrm{c}}})\in\widetilde{V_h(\Omega_{\mathrm{d}})}$, from Proposition 5.6.1 and Lemma 5.6.1, we obtain

$$\begin{aligned}
\|\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)} &\le C\|\mathbf{curl}(\boldsymbol{v}-\mathcal{E}_h(\boldsymbol{v}|_{\Omega_{\mathrm{c}}}))\|_{0,\Omega} + \|\mathcal{E}_h(\boldsymbol{v}|_{\Omega_{\mathrm{c}}})\|_{\mathbf{H}(\mathbf{curl};\Omega)}\\
&\le C\left(\|\boldsymbol{v}\|_{0,\Omega_{\mathrm{c}}} + \|\mathbf{curl}\,\boldsymbol{v}\|_{0,\Omega}\right).
\end{aligned} \tag{5.69}$$

Lemma 5.6.2 *If* $\boldsymbol{u} \in \mathrm{H}^1(0,T;\mathbf{H}(\mathbf{curl};\Omega))$, *then*

$$\begin{aligned}\sup_{t\in[0,T]} \|\boldsymbol{\delta}_h(t)\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} + \int_0^T \|\partial_t\boldsymbol{\delta}_h(s)\|^2_\sigma ds \\ \leq C\Bigg[\int_0^T \|\partial_t\boldsymbol{\rho}_h(t)\|^2_{\mathbf{H}(\mathbf{curl};\Omega)}dt + \sup_{t\in[0,T]}\|\mathbf{curl}\,\boldsymbol{\rho}_h(t)\|^2_{0,\Omega} \\ + \sup_{t\in[0,T]}\beta_h(\boldsymbol{u}(t))^2 + \int_0^T \beta_h(\partial_t\boldsymbol{u}(t))^2 dt\Bigg].\end{aligned} \tag{5.70}$$

Proof. We use similar arguments to those of [1, Lemma 5.7], but taking care of the terms arising from the boundary operators. Let $\boldsymbol{v} \in V_h(\Omega)$. A straightforward computation yields

$$\begin{aligned}&(\partial_t\boldsymbol{\delta}_h(t),\boldsymbol{v})_\sigma + a(\boldsymbol{\delta}_h(t),\boldsymbol{v}) + c_h(\boldsymbol{\delta}_h(t),\boldsymbol{v}) \\ &\quad = -(\partial_t\boldsymbol{\rho}_h(t),\boldsymbol{v})_\sigma - a(\boldsymbol{\rho}_h(t),\boldsymbol{v}) - c_h(\boldsymbol{\rho}_h(t),\boldsymbol{v}) + [c_h(\boldsymbol{u}(t),\boldsymbol{v}) - c(\boldsymbol{u}(t),\boldsymbol{v})].\end{aligned} \tag{5.71}$$

Then, from (5.69) it follows that

$$\begin{aligned}&(\partial_t\boldsymbol{\delta}_h(t),\boldsymbol{v})_\sigma + a(\boldsymbol{\delta}_h(t),\boldsymbol{v}) + c_h(\boldsymbol{\delta}_h(t),\boldsymbol{v}) \\ &\quad \leq \|\partial_t\boldsymbol{\rho}_h(t)\|_\sigma\|\boldsymbol{v}\|_\sigma + C\left(\|\boldsymbol{v}\|_{0,\Omega_c} + \|\mathbf{curl}\,\boldsymbol{v}\|_{0,\Omega}\right)\left[\|\boldsymbol{\rho}_h(t)\|_{\mathbf{H}(\mathbf{curl};\Omega)} + \beta_h(\boldsymbol{u}(t))\right] \\ &\quad \leq \frac{1}{2}\|\boldsymbol{v}\|^2_\sigma + \frac{1}{2\mu_1}\|\mathbf{curl}\,\boldsymbol{v}\|^2_{0,\Omega} + C\left[\|\partial_t\boldsymbol{\rho}_h(t)\|^2_\sigma + \|\boldsymbol{\rho}_h(t)\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} + \beta_h(\boldsymbol{u}(t))^2\right].\end{aligned}$$

Taking $\boldsymbol{v} = \boldsymbol{\delta}_h(t)$ in the last inequality and recalling that $c_h(\cdot,\cdot)$ is nonnegative, we deduce

$$\begin{aligned}&\frac{d}{dt}\|\boldsymbol{\delta}_h(t)\|^2_\sigma + \mu_1^{-1}\|\mathbf{curl}\,\boldsymbol{\delta}_h(t)\|_{0,\Omega} \\ &\qquad \leq \|\boldsymbol{\delta}_h(t)\|^2_\sigma + C\left[\|\partial_t\boldsymbol{\rho}_h(t)\|^2_\sigma + \|\boldsymbol{\rho}_h(t)\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} + \beta_h(\boldsymbol{u}(t))^2\right].\end{aligned}$$

We now integrate over $[0,T]$ (note that $\boldsymbol{\delta}_h(0) = \mathbf{0}$) and use the Gronwall's Inequality to obtain

$$\begin{aligned}&\|\boldsymbol{\delta}_h(t)\|^2_\sigma + \mu_1^{-1}\int_0^t \|\mathbf{curl}\,\boldsymbol{\delta}_h(t)\|^2_{0,\Omega}dt \\ &\qquad \leq C\int_0^T \left[\|\partial_t\boldsymbol{\rho}_h(t)\|^2_\sigma + \|\boldsymbol{\rho}_h(t)\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} + \beta_h(\boldsymbol{u}(t))^2\right]dt.\end{aligned}$$

Analogously, taking $\boldsymbol{v} = \partial_t\boldsymbol{\delta}_h(t)$ in (5.71) and using the linearity of $a(\cdot,\cdot)$, $c(\cdot,\cdot)$ and

$c_h(\cdot,\cdot)$, we deduce

$$\begin{aligned}&\|\partial_t\boldsymbol{\delta}_h(t)\|_\sigma^2+\frac{1}{2}\frac{d}{dt}\left[a(\boldsymbol{\delta}_h(t),\boldsymbol{\delta}_h(t))+c_h(\boldsymbol{\delta}_h(t),\boldsymbol{\delta}_h(t))\right]\\&=-(\partial_t\boldsymbol{\rho}_h(t),\partial_t\boldsymbol{\delta}_h(t))_\sigma-\frac{d}{dt}\left[a(\boldsymbol{\rho}_h(t),\boldsymbol{\delta}_h(t))+c_h(\boldsymbol{\rho}_h(t),\boldsymbol{\delta}_h(t))\right]+a(\partial_t\boldsymbol{\rho}_h(t),\boldsymbol{\delta}_h(t))\\&\quad+c_h(\partial_t\boldsymbol{\rho}_h(t),\boldsymbol{\delta}_h(t))+\frac{d}{dt}\left[c_h(\boldsymbol{u}(t),\boldsymbol{\delta}_h(t))-c(\boldsymbol{u}(t),\boldsymbol{\delta}_h(t))\right]\\&\quad+\left[c_h(\partial_t\boldsymbol{u}(t),\boldsymbol{v})-c(\partial_t\boldsymbol{u}(t),\boldsymbol{v})\right].\end{aligned}$$

Integrating over $[0,t]$ and using the Cauchy-Schwartz inequality, we have

$$\begin{aligned}&\int_0^t\|\partial_t\boldsymbol{\delta}_h(s)\|_\sigma^2ds+\|\mathbf{curl}\,\boldsymbol{\delta}_h(t)\|_{0,\Omega}^2\\&\qquad\le C\left[\int_0^t\|\mathbf{curl}\,\boldsymbol{\delta}_h(t)\|_{0,\Omega}+\int_0^T\|\partial_t\boldsymbol{\rho}_h(t)\|^2_{\mathbf{H}(\mathbf{curl};\Omega)}dt+\sup_{t\in[0,T]}\|\mathbf{curl}\,\boldsymbol{\rho}_h(t)\|_{0,\Omega}^2\right.\\&\qquad\qquad\left.+\beta_h(\boldsymbol{u}(t))^2+\int_0^T\beta_h(\partial_t\boldsymbol{u}(t))^2dt\right].\end{aligned}$$

Therefore, using Gronwall's Lemma, we conclude

$$\begin{aligned}&\int_0^t\|\partial_t\boldsymbol{\delta}_h(s)\|_\sigma^2ds+\|\mathbf{curl}\,\boldsymbol{\delta}_h(t)\|_{0,\Omega}^2\\&\le C\left[\int_0^T\|\partial_t\boldsymbol{\rho}_h(t)\|^2_{\mathbf{H}(\mathbf{curl};\Omega)}dt+\sup_{t\in[0,T]}\|\mathbf{curl}\,\boldsymbol{\rho}_h(t)\|_{0,\Omega}^2+\beta_h(\boldsymbol{u}(t))^2+\int_0^T\beta_h(\partial_t\boldsymbol{u}(t))^2dt\right].\end{aligned}$$

Finally, combining the last inequality with (5.6.1) and using (5.68), yield (5.70). □

Theorem 5.6.2 *Assume that* $\boldsymbol{u}\in\mathrm{H}^1(0,T;\mathbf{H}(\mathbf{curl};\Omega))$. *Let* $\boldsymbol{e}_h(t):=\boldsymbol{u}(t)-\boldsymbol{u}_h(t)$. *There exists* $C>0$, *such that*

$$\begin{aligned}&\sup_{t\in[0,T]}\|\boldsymbol{e}_h(t)\|^2_{\mathbf{H}(\mathbf{curl};\Omega)}+\int_0^T\|\boldsymbol{e}_h(t)\|^2_{\mathbf{H}(\mathbf{curl};\Omega)}\,dt+\int_0^T\|\partial_t\boldsymbol{e}_h(t)\|_\sigma^2\,dt\\&\qquad\le C\left\{\int_0^T\left[\inf_{\boldsymbol{v}\in X_h(\Omega)}\|\partial_t\boldsymbol{u}(t)-\boldsymbol{v}\|^2_{\mathbf{H}(\mathbf{curl};\Omega)}+\inf_{\chi\in\Lambda_h(\Gamma)}\|\partial_t\lambda(t)-\chi\|^2_{1/2,\Gamma}\right]dt\right.\\&\qquad\qquad\left.+\sup_{[0,T]}\inf_{\chi\in\Lambda_h(\Gamma)}\|\lambda(t)-\chi\|^2_{1/2,\Gamma}+\sup_{t\in[0,T]}\inf_{\boldsymbol{v}\in X_h(\Omega)}\|\boldsymbol{u}(t)-\boldsymbol{v}\|^2_{\mathbf{H}(\mathbf{curl};\Omega)}\right\}\end{aligned}\tag{5.72}$$

Proof. Since $\lambda(t)=R\mathrm{curl}_\Gamma\boldsymbol{K}^*\boldsymbol{\pi}_\tau\boldsymbol{u}(t)$, the regularity assumption on $\boldsymbol{u}$ implies

$$\lambda\in\mathrm{H}^1(0,T;\mathrm{H}_0^{1/2}(\Gamma))$$

and $\partial_t\lambda(t) = R\mathrm{curl}_\Gamma \boldsymbol{K}^*\boldsymbol{\pi}_\tau\partial_t\boldsymbol{u}(t)$. Consequently, from Corollary 5.2.1 and (5.59) it follows that

$$\beta_h(\boldsymbol{u}(t)) \leq C \inf_{\chi\in\Lambda_h(\Gamma)} \|\lambda(t)-\chi\|_{1/2,\Gamma}, \qquad \beta_h(\partial_t\boldsymbol{u}(t)) \leq C \inf_{\chi\in\Lambda_h(\Gamma)} \|\partial_t\lambda(t)-\chi\|_{1/2,\Gamma}. \quad (5.73)$$

Furthermore, since $\partial_t\Pi_h\boldsymbol{u}(t) = \Pi_h(\partial_t\boldsymbol{u}(t))$, the result follows by writing $\boldsymbol{e}_h(t) = \boldsymbol{\rho}_h(t) + \boldsymbol{\delta}_h(t)$ and using Lemma 5.6.2 and (5.66). □

For any $r \geq 0$, we consider the Sobolev space

$$\mathbf{H}^r(\mathbf{curl},\Omega_c) := \left\{\boldsymbol{v}\in \mathrm{H}^r(\Omega_c)^3 : \ \mathbf{curl}\,\boldsymbol{v}\in \mathrm{H}^r(\Omega_c)^3\right\}$$

endowed with the norm $\|\boldsymbol{v}\|^2_{\mathbf{H}^r(\mathbf{curl},\Omega_c)} := \|\boldsymbol{v}\|^2_{r,\Omega_c} + \|\mathbf{curl}\,\boldsymbol{v}\|^2_{r,\Omega_c}$ and analogously for Ω_d. It is well known that the Nédélec operator interpolation $\mathcal{I}_h\boldsymbol{v}\in X_h(\Omega_c)$ (and $\mathcal{I}_h\boldsymbol{w}\in X_h(\Omega_d)$) is well defined for any $\boldsymbol{v}\in\mathbf{H}^r(\mathbf{curl},\Omega_c)$ (resp. $\boldsymbol{w}\in\mathbf{H}^r(\mathbf{curl},\Omega_d)$) with $r > 1/2$; see, for instance, Lemma 5.1 of [11] or Lemma 4.7 of [14]. Moreover, if we introduce the space

$$\boldsymbol{\mathcal{X}} := \{\boldsymbol{v}\in\mathbf{H}(\mathbf{curl};\Omega) : \ \exists r > 1/2 \quad \boldsymbol{v}|_{\Omega_c}\in\mathbf{H}^r(\mathbf{curl},\Omega_c) \text{ and } \boldsymbol{v}|_{\Omega_d}\in\mathbf{H}^r(\mathbf{curl},\Omega_d)\}$$

endowed with the $\mathbf{H}^r(\mathbf{curl},\Omega_c)\times\mathbf{H}^r(\mathbf{curl},\Omega_d)$-norm, then $\mathcal{I}_h : \boldsymbol{\mathcal{X}}\to X_h(\Omega)$ is well defined, bounded uniformly in h and the following interpolation error estimate holds true (see Lemma 5.1 of [16] or Proposition 5.6 of [11]):

$$\|\boldsymbol{v}-\mathcal{I}_h\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)} \leq Ch^{\min\{r,m\}}\|\boldsymbol{v}\|_{\boldsymbol{\mathcal{X}}} \qquad \forall\boldsymbol{v}\in\mathbf{H}^r(\mathbf{curl},\Omega). \quad (5.74)$$

Lemma 5.6.3 *Let* $(\boldsymbol{u},p,\lambda)$ *be the solution of* (5.35). *If we assume that*

$$\boldsymbol{u}\in \mathrm{H}^1(0,T;\boldsymbol{\mathcal{X}}) \quad \textit{and} \quad \mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}\in\mathrm{H}^1(0,T;\boldsymbol{\mathcal{X}}),$$

then

$$\inf_{\chi\in\Lambda_h(\Gamma)}\|\lambda(t)-\chi\|_{1/2,\Gamma} \leq Ch^{\min\{r,m\}}\|\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}(t)\|_{\boldsymbol{\mathcal{X}}} \quad (5.75)$$

and

$$\inf_{\chi\in\Lambda_h(\Gamma)}\|\partial_t\lambda(t)-\chi\|_{1/2,\Gamma} \leq Ch^{\min\{r,m\}}\|\partial_t(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}(t))\|_{\boldsymbol{\mathcal{X}}}. \quad (5.76)$$

Proof. Let $\mathcal{I}_h^\Gamma$ be the $2D$ Nédélec interpolant on $\mathcal{T}_h(\Gamma)$. We recall that this interpolant relates to the standard $3D$ Nédélec interpolant as follows

$$\boldsymbol{\pi}_\tau(\mathcal{I}_h\boldsymbol{v}) = \mathcal{I}_h^\Gamma(\boldsymbol{\pi}_\tau\boldsymbol{v}),$$

as far as $\boldsymbol{v}$ is smooth enough for the interpolant to be will defined. Consequently, since $\mathbf{curl}_\Gamma \lambda = \boldsymbol{\gamma}_\tau(\mu^{-1} \mathbf{curl}\, \boldsymbol{u})$, we obtain

$$\boldsymbol{\pi}_\tau(\mathcal{I}_h(\mu^{-1} \mathbf{curl}\, \boldsymbol{u})) = \mathcal{I}_h^\Gamma(\boldsymbol{\pi}_\tau(\mu^{-1} \mathbf{curl}\, \boldsymbol{u})) = \mathcal{I}_h^\Gamma(\boldsymbol{n} \times \boldsymbol{\gamma}_\tau(\mu^{-1} \mathbf{curl}\, \boldsymbol{u}))$$
$$= \mathcal{I}_h^\Gamma(\boldsymbol{n} \times \mathbf{curl}_\Gamma \lambda) = \mathcal{I}_h^\Gamma(\mathbf{grad}_\Gamma \lambda).$$

Therefore, a similar argument to that used in the proof of Proposition 5.6.1 allow us to show that there exists $\chi(t) \in \Lambda_h(\Gamma)$ such that $\boldsymbol{\pi}_\tau(\mathcal{I}_h(\mu^{-1} \mathbf{curl}\, \boldsymbol{u}(t))) = \mathbf{grad}_\Gamma \chi(t)$, or equivalently

$$\boldsymbol{\gamma}_\tau(\mathcal{I}_h(\mu^{-1} \mathbf{curl}\, \boldsymbol{u}(t))) = \mathbf{curl}_\Gamma \chi(t).$$

Then, using Corollary 5.2.1, we deduce

$$\begin{aligned}
\inf_{\chi \in \Lambda_h(\Gamma)} \|\lambda(t) - \chi\|_{1/2,\Gamma} &\leq C_1 \inf_{\chi \in \Lambda_h(\Gamma)} \|\mathbf{curl}_\Gamma \lambda(t) - \mathbf{curl}_\Gamma \chi\|_{-1/2,\Gamma} \\
&\leq C_1 \|\mathbf{curl}_\Gamma \lambda(t) - \boldsymbol{\gamma}_\tau \mathcal{I}_h(\mu^{-1} \mathbf{curl}\, \boldsymbol{u}(t))\|_{-1/2,\Gamma} \\
&= C_1 \|\boldsymbol{\gamma}_\tau(\mathbf{I}_\mathrm{d} - \mathcal{I}_h)(\mu^{-1} \mathbf{curl}\, \boldsymbol{u}(t))\|_{-1/2,\Gamma} \\
&\leq C_2 \|(\mathbf{I}_\mathrm{d} - \mathcal{I}_h)(\mu^{-1} \mathbf{curl}\, \boldsymbol{u}(t))\|_{\mathbf{H}(\mathbf{curl};\Omega)}.
\end{aligned}$$

Hence, (5.75) follows by using the interpolation error estimate (5.74).

Finally, proceeding as above, the regularity of $\mu^{-1} \mathbf{curl}\, \boldsymbol{u}$ ensures that

$$\boldsymbol{\pi}_\tau(\mathcal{I}_h(\partial_t(\mu^{-1} \mathbf{curl}\, \boldsymbol{u}))) = \mathcal{I}_h^\Gamma(\mathbf{grad}_\Gamma \partial_t \lambda).$$

Consequently, a similar analysis to the previous one, shows (5.76). □

Corollary 5.6.1 *Under the assumptions of Lemma 5.6.3, there holds*

$$\begin{aligned}
&\sup_{t \in [0,T]} \|\boldsymbol{e}_h(t)\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} + \int_0^T \|\boldsymbol{e}_h(t)\|^2_{\mathbf{H}(\mathbf{curl};\Omega)}\, dt + \int_0^T \|\partial_t \boldsymbol{e}_h(t)\|^2_\sigma\, dt \\
&\quad \leq Ch^{2l} \Bigg\{ \sup_{t \in [0,T]} \|\boldsymbol{u}(t)\|^2_{\boldsymbol{\mathcal{X}}} + \sup_{t \in [0,T]} \|\mu^{-1} \mathbf{curl}\, \boldsymbol{u}(t)\|^2_{\boldsymbol{\mathcal{X}}} \\
&\qquad + \int_0^T \|\partial_t \boldsymbol{u}(t)\|^2_{\boldsymbol{\mathcal{X}}}\, dt + \int_0^T \|\partial_t(\mu^{-1} \mathbf{curl}\, \boldsymbol{u}(t))\|^2_{\boldsymbol{\mathcal{X}}}\, dt \Bigg\},
\end{aligned}$$

with $l := \min\{r, m\}$.

Proof. It is a direct consequence of Theorem 5.6.2, Lemma 5.6.3 and the interpolation error estimate (3.48). □

Remark 5.6.2 *Let us recall that*

$$\lambda(t) = \mu_0^{-1} R(\mathrm{curl}_\Gamma \boldsymbol{K}^* \boldsymbol{\pi}_\tau \boldsymbol{u}(t)) \quad \textit{and} \quad \lambda_h(t) = \mu_0^{-1} R_h(\mathrm{curl}_\Gamma \boldsymbol{K}^* \boldsymbol{\pi}_\tau \boldsymbol{u}_h(t)).$$

Therefore, using (5.73) *and the uniform boundedness of* R_h*, we obtain*

$$\begin{aligned}\mu_0 \|\lambda(t) - \lambda_h(t)\|_{1/2,\Gamma} &\leq \beta_h(\boldsymbol{u}(t)) + \|R_h \mathrm{curl}_\Gamma \boldsymbol{K}^* \boldsymbol{\pi}_\tau (\boldsymbol{u} - \boldsymbol{u}_h)(t)\|_{1/2,\Gamma} \\ &\leq C \left\{ \inf_{\chi \in \Lambda_h(\Gamma)} \|\lambda(t) - \chi\|_{1/2,\Gamma} + \|\boldsymbol{e}_h(t)\|_{\mathbf{H}(\mathbf{curl};\Omega)} \right\}.\end{aligned}$$

Consequently, using Lemma 5.6.3 and Corollary 5.6.1 we have

$$\int_0^T \|\lambda(t) - \lambda_h(t)\|_{1/2,\Gamma}^2 dt \leq C h^{2l},$$

with $l := \min\{r, m\}$.

5.7 Analysis of a fully-discrete scheme.

We consider a uniform partition $\{t_n := n\Delta t : \ n = 0, \ldots, N\}$ of $[0, T]$ with a step size $\Delta t := \frac{T}{N}$. For any finite sequence $\{\theta^n : \ n = 0, \cdots, N\}$, let

$$\bar{\partial}\theta^n := \frac{\theta^n - \theta^{n-1}}{\Delta t}, \qquad n = 1, 2, \ldots, N.$$

A fully-discrete version of problem (5.35) reads as follows:
Find $(\boldsymbol{u}_h^n, p_h^n, \lambda_h^n) \in X_h(\Omega) \times M_h(\Omega_\mathrm{d}) \times \Lambda_h(\Gamma)$, $n = 1, \cdots, N$, such that

$$\begin{aligned}
(\bar{\partial}\boldsymbol{u}_h^n, \boldsymbol{v})_\sigma + b(\boldsymbol{v}, \bar{\partial}p_h^n) + a(\boldsymbol{u}_h^n, \boldsymbol{v}) & \\
+ \mu_0^{-1} \langle S(\mathrm{curl}_\Gamma \boldsymbol{\pi}_\tau \boldsymbol{u}_h^n), \mathrm{curl}_\Gamma \boldsymbol{\pi}_\tau \boldsymbol{v} \rangle_{1/2,\Gamma} & \\
+ \langle \boldsymbol{K}\, \mathbf{curl}_\Gamma\, \lambda_h^n(t), \boldsymbol{\pi}_\tau \boldsymbol{v} \rangle_{\tau,\Gamma} &= (\boldsymbol{f}(t_n), \boldsymbol{v})_{0,\Omega} \qquad \forall \boldsymbol{v} \in X_h(\Omega), \\
- \langle \mathbf{curl}_\Gamma\, \eta, \boldsymbol{V}(\mathbf{curl}_\Gamma\, \lambda_h^n) \rangle_{\tau,\Gamma} + \mu_0^{-1} \langle \boldsymbol{K}(\mathbf{curl}_\Gamma\, \eta), \boldsymbol{\pi}_\tau \boldsymbol{u}_h^n \rangle_{\tau,\Gamma} &= 0 \qquad \forall \eta \in \mathrm{H}_0^{1/2}(\Gamma), \\
b(\boldsymbol{u}_h^n, q) &= 0 \qquad \forall q \in M_h(\Omega_\mathrm{d}), \\
\boldsymbol{u}_h^0|_{\Omega_\mathrm{c}} &= \boldsymbol{0}, \\
p_h^0 &= 0, \\
\lambda_h^0 &= 0.
\end{aligned} \tag{5.77}$$

The second equation of (5.77) means $\lambda_h^n = \mu_0^{-1} R_h(\mathrm{curl}_\Gamma \boldsymbol{K}^* \boldsymbol{\pi}_\tau \boldsymbol{u}_h^n)$. Consequently, for the analysis, we can eliminate λ_h^n in (5.77) to obtain the problem:

Find $(\boldsymbol{u}_h^n, p_h^n) \in X_h(\Omega) \times M_h(\Omega_{\mathrm{d}}),\ n = 1, \cdots, N$, such that

$$\begin{aligned} (\bar{\partial}\boldsymbol{u}_h^n, \boldsymbol{v})_\sigma + b(\boldsymbol{v}, \bar{\partial}p_h^n) + a(\boldsymbol{u}_h^n, \boldsymbol{v}) + c_h(\boldsymbol{u}_h^n, \boldsymbol{v}) = (\boldsymbol{f}(t_n), \boldsymbol{v})_{0,\Omega} \quad & \forall \boldsymbol{v} \in X_h(\Omega), \\ b(\boldsymbol{u}_h^n, q) = 0 \quad & \forall q \in M_h(\Omega_{\mathrm{d}}), \\ \boldsymbol{u}_h^0|_{\Omega_{\mathrm{c}}} = \mathbf{0}, & \\ p_h^0 = 0. & \end{aligned} \tag{5.78}$$

Hence, at each iteration step we have to find $(\boldsymbol{u}_h^n, p_h^n) \in X_h(\Omega) \times M_h(\Omega_{\mathrm{d}})$ such that

$$\begin{aligned} (\boldsymbol{u}_h^n, \boldsymbol{v})_\sigma + \Delta t\, [a(\boldsymbol{u}_h^n, \boldsymbol{v}) + c_h(\boldsymbol{u}_h^n, \boldsymbol{v})] + b(\boldsymbol{v}, p_h^n) = F_n(\boldsymbol{v}) \qquad & \forall \boldsymbol{v} \in X_h(\Omega), \\ b(\boldsymbol{u}_h^n, q) = 0 \qquad & \forall q \in M_h(\Omega_{\mathrm{d}}), \end{aligned}$$

where

$$F_n(\boldsymbol{v}) := \Delta t(\boldsymbol{f}(t_n), \boldsymbol{v})_{0,\Omega} + (\boldsymbol{u}_h^{n-1}, \boldsymbol{v})_\sigma + b(\boldsymbol{v}, p_h^{n-1}).$$

The existence and uniqueness of $(\boldsymbol{u}_h^n, \lambda_h^n)$ is a direct consequence of the Babuška-Brezzi theory. Indeed, the bilinear form b satisfies the discrete inf-sup condition (5.62) and the inner product

$$(\boldsymbol{v}, \boldsymbol{w}) \mapsto (\boldsymbol{v}, \boldsymbol{w})_\sigma + \Delta t\, [a(\boldsymbol{v}, \boldsymbol{w}) + c_h(\boldsymbol{v}, \boldsymbol{w})]$$

induces a norm on its kernel $V_h(\Omega)$ (cf. Lemma 5.6.1).

5.7.1 Error estimates.

Lemma 5.7.1 *Let* $\boldsymbol{\rho}^n := \boldsymbol{u}(t_n) - \Pi_h \boldsymbol{u}(t_n)$, $\boldsymbol{\delta}^n := \Pi_h \boldsymbol{u}(t_n) - \boldsymbol{u}_h^n$, $\boldsymbol{\tau}^n := \bar{\partial}\boldsymbol{u}(t_n) - \partial_t \boldsymbol{u}(t_n)$ *and* β_h *as defined in* (5.67). *There exists* $C > 0$ *independent of* h *and* Δt *such that*

$$\begin{aligned} & \max_{1 \le k \le n} \|\boldsymbol{\delta}^k\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} + \Delta t \sum_{k=1}^n \|\bar{\partial}\boldsymbol{\delta}^k\|^2_\sigma \\ & \le C \left\{ \Delta t \sum_{k=1}^n \left[\|\bar{\partial}\boldsymbol{\rho}^k\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} + \|\boldsymbol{\tau}^k\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} + \beta_h(\partial_t \boldsymbol{u}(t_k))^2 \right] \right. \\ & \qquad \left. + \max_{1 \le k \le n} \|\boldsymbol{\rho}^k\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} + \max_{1 \le k \le n} \beta_h(\boldsymbol{u}(t_k))^2 \right\}. \end{aligned} \tag{5.79}$$

Proof. It is straightforward to show that

$$\begin{aligned} & (\bar{\partial}\boldsymbol{\delta}^k, \boldsymbol{v})_\sigma + a(\boldsymbol{\delta}^k, \boldsymbol{v}) + c_h(\boldsymbol{\delta}^k, \boldsymbol{v}) \\ & \quad = -(\bar{\partial}\boldsymbol{\rho}^k, \boldsymbol{v})_\sigma - a(\boldsymbol{\rho}^k, \boldsymbol{v}) + (\boldsymbol{\tau}^k, \boldsymbol{v})_\sigma + c_h(\boldsymbol{u}(t_k), \boldsymbol{v}) - c(\boldsymbol{u}(t_k), \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in V_h(\Omega). \end{aligned} \tag{5.80}$$

Choosing $\boldsymbol{v} = \boldsymbol{\delta}^k$ in the last identity, recalling that $c_h(\cdot,\cdot)$ is nonnegative and using the estimates

$$a(\boldsymbol{\delta}^k, \boldsymbol{\delta}^k) \geq \mu_1^{-1} \| \mathbf{curl}\, \boldsymbol{\delta}^k \|_{0,\Omega}^2 \quad \text{and} \quad (\bar{\partial}\boldsymbol{\delta}^k, \boldsymbol{\delta}^k)_\sigma \geq \frac{1}{2\Delta t} \left(\|\boldsymbol{\delta}^k\|_\sigma^2 - \|\boldsymbol{\delta}^{k-1}\|_\sigma^2 \right),$$

together with

$$\|\boldsymbol{\delta}^k\|_{\mathbf{H}(\mathbf{curl};\Omega)} \leq C \left[\|\boldsymbol{\delta}^k\|_\sigma + \|\mathbf{curl}\, \boldsymbol{\delta}^k\|_{0,\Omega} \right] \qquad k = 1, \ldots, n \tag{5.81}$$

(which is a particular case of (5.69)) and the Cauchy-Schwartz inequality, yield

$$\begin{aligned} &\|\boldsymbol{\delta}^k\|_\sigma^2 - \|\boldsymbol{\delta}^{k-1}\|_\sigma^2 + \Delta t\, \mu_1^{-1} \| \mathbf{curl}\, \boldsymbol{\delta}^k \|_{0,\Omega}^2 \\ &\qquad \leq \frac{\Delta t}{2T} \|\boldsymbol{\delta}^k\|_\sigma^2 + C\Delta t \left[\|\bar{\partial}\boldsymbol{\rho}^k\|_\sigma^2 + \| \mathbf{curl}\, \boldsymbol{\rho}^k \|_{0,\Omega} + \|\boldsymbol{\tau}^k\|_\sigma^2 + \beta_h(\boldsymbol{u}(t_k))^2 \right]. \end{aligned} \tag{5.82}$$

As a consequence,

$$\begin{aligned} &\|\boldsymbol{\delta}^k\|_\sigma^2 - \|\boldsymbol{\delta}^{k-1}\|_\sigma^2 \\ &\qquad \leq \frac{\Delta t}{2T} \|\boldsymbol{\delta}^k\|_\sigma^2 + C\Delta t \left[\|\bar{\partial}\boldsymbol{\rho}^k\|_\sigma^2 + \| \mathbf{curl}\, \boldsymbol{\rho}^k \|_{0,\Omega} + \|\boldsymbol{\tau}^k\|_\sigma^2 + \beta_h(\boldsymbol{u}(t_k))^2 \right]. \end{aligned}$$

Then, summing over k and using the discrete Gronwall's Lemma ([62, Lemma 1.4.2]) and the fact that $\boldsymbol{\delta}^0 = \mathbf{0}$, lead to

$$\|\boldsymbol{\delta}^n\|_\sigma^2 \leq C\Delta t \sum_{k=1}^n \left(\|\bar{\partial}\boldsymbol{\rho}^k\|_\sigma^2 + \| \mathbf{curl}\, \boldsymbol{\rho}^k \|_{0,\Omega}^2 + \|\boldsymbol{\tau}^k\|_\sigma^2 + \beta_h(\boldsymbol{u}(t_k))^2 \right), \tag{5.83}$$

for $n = 1, \ldots, N$. Inserting the last inequality in (5.82) and summing over k we have the estimate

$$\begin{aligned} &\|\boldsymbol{\delta}^n\|_\sigma^2 + \Delta t \sum_{k=1}^n \| \mathbf{curl}\, \boldsymbol{\delta}^k \|_{0,\Omega}^2 \\ &\qquad \leq C\Delta t \left(\sum_{k=1}^n \|\bar{\partial}\boldsymbol{\rho}^k\|_\sigma^2 + \sum_{k=1}^n \| \mathbf{curl}\, \boldsymbol{\rho}^k \|_{0,\Omega}^2 + \sum_{k=1}^n \|\boldsymbol{\tau}^k\|_\sigma^2 + \sum_{k=1}^n \beta_h(\boldsymbol{u}(t_k))^2 \right). \end{aligned} \tag{5.84}$$

Let us now take $\boldsymbol{v} = \bar{\partial}\boldsymbol{\delta}^k$ in (5.80):

$$\begin{aligned} &\|\bar{\partial}\boldsymbol{\delta}^k\|_\sigma^2 + a(\boldsymbol{\delta}^k, \bar{\partial}\boldsymbol{\delta}^k) + c_h(\boldsymbol{\delta}^k, \bar{\partial}\boldsymbol{\delta}^k) \\ &\qquad = -(\bar{\partial}\boldsymbol{\rho}^k, \bar{\partial}\boldsymbol{\delta}^k)_\sigma + (\boldsymbol{\tau}^k, \bar{\partial}\boldsymbol{\delta}^k)_\sigma - a(\boldsymbol{\rho}^k, \bar{\partial}\boldsymbol{\delta}^k) + c_h(\boldsymbol{u}(t_k), \bar{\partial}\boldsymbol{\delta}^k) - c(\boldsymbol{u}(t_k), \bar{\partial}\boldsymbol{\delta}^k) \\ &\qquad = -(\bar{\partial}\boldsymbol{\rho}^k, \bar{\partial}\boldsymbol{\delta}^k)_\sigma + (\boldsymbol{\tau}^k, \bar{\partial}\boldsymbol{\delta}^k)_\sigma + a(\bar{\partial}\boldsymbol{\rho}^k, \boldsymbol{\delta}^{k-1}) - c(\bar{\partial}\boldsymbol{u}(t_k), \boldsymbol{\delta}^{k-1}) \\ &\qquad\quad + c_h(\bar{\partial}\boldsymbol{u}(t_k), \boldsymbol{\delta}^{k-1}) - \frac{1}{\Delta t}(\gamma_k - \gamma_{k-1}), \end{aligned}$$

where

$$\gamma_k := a(\boldsymbol{\rho}^k, \boldsymbol{\delta}^k) - c(\boldsymbol{u}(t_k), \boldsymbol{\delta}^k) + c_h(\boldsymbol{u}(t_k), \boldsymbol{\delta}^k).$$

Consequently,

$$\begin{aligned}
&\|\bar{\partial}\boldsymbol{\delta}^k\|_\sigma^2 + a(\boldsymbol{\delta}^k, \bar{\partial}\boldsymbol{\delta}^k) + c_h(\boldsymbol{\delta}^k, \bar{\partial}\boldsymbol{\delta}^k) \\
&\qquad = -(\bar{\partial}\boldsymbol{\rho}^k, \bar{\partial}\boldsymbol{\delta}^k)_\sigma + (\boldsymbol{\tau}^k, \bar{\partial}\boldsymbol{\delta}^k)_\sigma + a(\bar{\partial}\boldsymbol{\rho}^k, \boldsymbol{\delta}^{k-1}) - c(\boldsymbol{\tau}^k, \boldsymbol{\delta}^{k-1}) + c_h(\boldsymbol{\tau}^k, \boldsymbol{\delta}^{k-1}) \\
&\qquad\quad - c(\partial_t \boldsymbol{u}(t_k), \boldsymbol{\delta}^{k-1}) + c_h(\partial_t \boldsymbol{u}(t_k), \boldsymbol{\delta}^{k-1}) - \frac{1}{\Delta t}(\gamma_k - \gamma_{k-1}).
\end{aligned} \tag{5.85}$$

On the other hand, since the bilinear forms $a(\cdot,\cdot)$ and $c_h(\cdot,\cdot)$ are nonnegative, we have that

$$a(\boldsymbol{\delta}^k, \bar{\partial}\boldsymbol{\delta}^k) \geq \frac{1}{2\Delta t}\left[a(\boldsymbol{\delta}^k, \boldsymbol{\delta}^k) - a(\boldsymbol{\delta}^{k-1}, \boldsymbol{\delta}^{k-1})\right]$$

and

$$c_h(\boldsymbol{\delta}^k, \bar{\partial}\boldsymbol{\delta}^k) \geq \frac{1}{2\Delta t}\left[c_h(\boldsymbol{\delta}^k, \boldsymbol{\delta}^k) - c_h(\boldsymbol{\delta}^{k-1}, \boldsymbol{\delta}^{k-1})\right].$$

Hence, using these inequalities in (5.85), the Cauchy-Schwartz inequality leads to

$$\begin{aligned}
&\frac{1}{2}\|\bar{\partial}\boldsymbol{\delta}^k\|_\sigma^2 + \frac{1}{2\Delta t}\left[a(\boldsymbol{\delta}^k, \boldsymbol{\delta}^k) - a(\boldsymbol{\delta}^{k-1}, \boldsymbol{\delta}^{k-1})\right] + \frac{1}{2\Delta t}\left[c_h(\boldsymbol{\delta}^k, \boldsymbol{\delta}^k) - c_h(\boldsymbol{\delta}^{k-1}, \boldsymbol{\delta}^{k-1})\right] \\
&\quad \leq C\left(\|\bar{\partial}\boldsymbol{\rho}^k\|_\sigma^2 + \|\boldsymbol{\tau}^k\|_\sigma^2\right) + a(\bar{\partial}\boldsymbol{\rho}^k, \boldsymbol{\delta}^{k-1}) - c(\boldsymbol{\tau}^k, \boldsymbol{\delta}^{k-1}) + c_h(\boldsymbol{\tau}^k, \boldsymbol{\delta}^{k-1}) - c(\partial_t \boldsymbol{u}(t_k), \boldsymbol{\delta}^{k-1}) \\
&\qquad + c_h(\partial_t \boldsymbol{u}(t_k), \boldsymbol{\delta}^{k-1}) - \frac{1}{\Delta t}(\gamma_k - \gamma_{k-1}),
\end{aligned}$$

then, summing over k and recalling that $c_h(\cdot,\cdot)$ is nonnegative, we deduce

$$\begin{aligned}
&\frac{1}{2}\sum_{k=1}^{n}\|\bar{\partial}\boldsymbol{\delta}^k\|_\sigma^2 + \frac{1}{2\mu_1\Delta t}\|\mathbf{curl}\,\boldsymbol{\delta}^n\|_{0,\Omega}^2 \\
&\qquad \leq C\sum_{k=1}^{n}\left(\|\bar{\partial}\boldsymbol{\rho}^k\|_\sigma^2 + \|\boldsymbol{\tau}^k\|_\sigma^2\right) + \sum_{k=1}^{n}(\theta_{1,k} + \theta_{2,k} + \theta_{3,k}) + \frac{1}{\Delta t}|\gamma_n|,
\end{aligned} \tag{5.86}$$

where

$$\begin{aligned}
\theta_{1,k} &:= |a(\bar{\partial}\boldsymbol{\rho}^k, \boldsymbol{\delta}^{k-1})|, \\
\theta_{2,k} &:= |c(\boldsymbol{\tau}^k, \boldsymbol{\delta}^{k-1}) - c_h(\boldsymbol{\tau}^k, \boldsymbol{\delta}^{k-1})|, \\
\theta_{3,k} &:= |c(\partial_t \boldsymbol{u}(t_k), \boldsymbol{\delta}^{k-1}) - c_h(\partial_t \boldsymbol{u}(t_k), \boldsymbol{\delta}^{k-1})|.
\end{aligned}$$

Using the Cauchy-Schwartz inequality and (5.81), we obtain

$$\sum_{k=1}^{n}\theta_{1,k} \leq \sum_{k=1}^{n}\|\mathbf{curl}\,\boldsymbol{\delta}^{k-1}\|_{0,\Omega}^2 + \frac{1}{4\mu_0}\sum_{k=1}^{n}\|\mathbf{curl}\,\bar{\partial}\boldsymbol{\rho}^k\|_{0,\Omega}^2,$$
$$\sum_{k=1}^{n}\theta_{2,k} \leq \sum_{k=1}^{n}\left[\|\boldsymbol{\delta}^{k-1}\|_{\sigma}^2 + \|\mathbf{curl}\,\boldsymbol{\delta}^{k-1}\|_{0,\Omega}^2 + C\|\boldsymbol{\tau}^k\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2\right],$$
$$\sum_{k=1}^{n}\theta_{3,k} \leq \sum_{k=1}^{n}\left[\|\boldsymbol{\delta}^{k-1}\|_{\sigma}^2 + \|\mathbf{curl}\,\boldsymbol{\delta}^{k-1}\|_{0,\Omega}^2 + C\beta_h(\partial_t\boldsymbol{u}(t_n))^2\right],$$
$$|\gamma_n| \leq \|\boldsymbol{\delta}^n\|_{\sigma}^2 + \frac{1}{4\mu_1}\|\mathbf{curl}\,\boldsymbol{\delta}^n\|_{0,\Omega}^2 + C\left[\|\mathbf{curl}\,\boldsymbol{\rho}^n\|_{0,\Omega}^2 + \beta_h(\boldsymbol{u}(t_n))^2\right].$$

Substituting the last inequalities in (5.86) and using (5.84), we obtain

$$\begin{aligned}\Delta t\sum_{k=1}^{n}\|\bar{\partial}\boldsymbol{\delta}^k\|_{\sigma}^2 &+ \|\mathbf{curl}\,\boldsymbol{\delta}^n\|_{0,\Omega}^2\\ &\leq C\left\{\Delta t\sum_{k=1}^{n}\left[\|\bar{\partial}\boldsymbol{\rho}^k\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2 + \|\boldsymbol{\rho}^k\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2 + \|\boldsymbol{\tau}^k\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2 + \beta_h(\boldsymbol{u}(t_k))^2\right.\right.\\ &\left.\left.+\beta_h(\partial_t\boldsymbol{u}(t_k))^2\right] + \|\mathbf{curl}\,\boldsymbol{\rho}^n\|_{0,\Omega}^2 + \beta_h(\boldsymbol{u}(t_n))^2\right\}.\end{aligned}$$

Combining this last inequality with (5.84) and (5.81), we conclude (5.79). □

Theorem 5.7.1 *Assume that* $\boldsymbol{u} \in \mathrm{H}^2(0,T;\boldsymbol{\mathcal{X}})$ *and let* $\boldsymbol{e}^n := \boldsymbol{u}(t_n) - \boldsymbol{u}_h^n$. *Then, there exists a constant* $C > 0$, *independent of* h *and* Δt, *such that*

$$\begin{aligned}\max_{1\leq n\leq N}\|\boldsymbol{e}^n\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2 &+ \Delta t\sum_{k=1}^{N}\|\bar{\partial}\boldsymbol{e}^k\|_{\sigma}^2\\ &\leq C\left\{\max_{1\leq n\leq N}\inf_{\boldsymbol{v}\in X_h(\Omega)}\|\boldsymbol{u}(t_n)-\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2 + \max_{1\leq n\leq N}\inf_{\xi\in\Lambda_h(\Gamma)}\|\lambda(t_n)-\xi\|_{1/2,\Gamma}^2\right.\\ &+\Delta t\sum_{n=1}^{N}\inf_{\xi\in\Lambda_h(\Gamma)}\|\partial_t\lambda(t_n)-\eta\|_{1/2,\Gamma}^2 + \int_0^T\left(\inf_{\boldsymbol{v}\in X_h(\Omega)}\|\partial_t\boldsymbol{u}(t)-\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2\right)dt\\ &\left.+(\Delta t)^2\int_0^T\|\partial_{tt}\boldsymbol{u}(t)\|_{\mathbf{H}(\mathbf{curl};\Omega)}^2\,dt\right\}.\end{aligned}$$

Proof. A Taylor expansion shows that

$$\bar{\partial}\boldsymbol{u}(t_k) = \partial_t\boldsymbol{u}(t_k) + \frac{1}{\Delta t}\int_{t_{k-1}}^{t_k}(t_{k-1}-t)\partial_{tt}\boldsymbol{u}(t)\,dt.$$

Consequently,

$$\sum_{k=1}^{n} \|\boldsymbol{\tau}^k\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} \leq \Delta t \int_0^T \|\partial_{tt}\boldsymbol{u}(t)\|^2_{\mathbf{H}(\mathbf{curl};\Omega)}\, dt.$$

Moreover, noticing that $\boldsymbol{\rho}^k = \boldsymbol{\rho}_h(t_k)$ and using (5.66), we have

$$\begin{aligned}\sum_{k=1}^{n} \|\bar{\partial}\boldsymbol{\rho}^k\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} \leq& \frac{1}{\Delta t}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} \|\partial_t\boldsymbol{\rho}_h(t)\|^2_{\mathbf{H}(\mathbf{curl};\Omega)}\, dt \\ \leq& \frac{1}{\Delta t}\int_0^T \left(\inf_{\boldsymbol{v}\in X_h(\Omega)} \|\partial_t\boldsymbol{u}(t) - \boldsymbol{v}\|^2_{\mathbf{H}(\mathbf{curl};\Omega)}\right) dt.\end{aligned}$$

Therefore, writing $\boldsymbol{e}^n = \boldsymbol{\delta}^n + \boldsymbol{\rho}^n$ and using (5.73), (5.66) and Lemma 5.6.2, we conclude the result. □

Using Lemma 5.6.3, Theorem 5.7.1 and the interpolation error estimate (5.74), we deduce the following result.

Corollary 5.7.1 *Under the assumptions of Lemma 5.6.3 and Theorem 5.7.1, there holds*

$$\begin{aligned}&\max_{1\leq n\leq N} \|\boldsymbol{e}^n\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} + \Delta t \sum_{k=1}^{N} \|\bar{\partial}\boldsymbol{e}^k\|^2_{\sigma} \\ &\quad\leq Ch^{2l}\Bigg\{\max_{1\leq n\leq N}\|\boldsymbol{u}(t_n)\|^2_{\boldsymbol{\mathcal{X}}} + \max_{1\leq n\leq N}\|\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}(t_n)\|^2_{\boldsymbol{\mathcal{X}}} \\ &\qquad\qquad + \max_{1\leq n\leq N}\|\partial_t(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}(t_n))\|^2_{\boldsymbol{\mathcal{X}}} + \int_0^T \|\partial_t\boldsymbol{u}(t)\|^2_{\boldsymbol{\mathcal{X}}}\, dt\Bigg\} \\ &\quad\quad + C(\Delta t)^2\int_0^T \|\partial_{tt}\boldsymbol{u}(t)\|^2_{\sigma}\, dt,\end{aligned}$$

with $l := \min\{m, r\}$.

Remark 5.7.1 *Since* $\lambda_h^n = \mu_0^{-1}R_h(\mathrm{curl}_\Gamma\boldsymbol{K}^*\boldsymbol{\pi}_\tau\boldsymbol{u}_h^n))$, *we proceed as in Remark 5.6.2 to obtain*

$$\Delta t\sum_{k=1}^{n}\|\lambda(t_n) - \lambda_h^n\|^2_{1/2,\Gamma} \leq C[h^{2l} + (\Delta t)^2],$$

with $l := \min\{r, m\}$.

Chapter 6

Conclusiones y trabajo futuro

En este capítulo se presenta un resumen de los principales aportes de esta tesis y una descripción del trabajo futuro a desarrollar.

6.1 Conclusiones

1. Se demostró que la formulación $\boldsymbol{A}, V - \boldsymbol{A} - \psi$ del problema de corrientes inducidas en un dominio acotado es un problema bien planteado y que su discretización a través de elementos finitos usuales genera una aproximación numérica de orden óptimo. Esto proporciona el soporte matemático de la conocida eficiencia de esta aproximación.

2. En el estudio de la formulación $\boldsymbol{A}, V - \boldsymbol{A} - \psi$ se muestra la necesidad de que el dominio del potencial $\boldsymbol{A}$ tenga componentes conexas convexas. Debido a que este dominio puede escogerse libremente (solo debe contener al conductor y a la fuente de corriente), esta condición no genera ningún inconveniente en la práctica.

3. Se propuso una formulación para un problema evolutivo de corrientes inducidas en un dominio acotado. Las variables de esta formulación son una primitiva del campo eléctrico y un multiplicador de Lagrange que impone las condiciones de divergencia nula en el material dieléctrico. Se demostró que esta formulación está bien planteada y que el multiplicador de Lagrange es identicamente nulo.

4. Se verificó que la formulación mencionada en el punto anterior, puede extenderse para estudiar modelos que consideran materiales conductores ferromagnéticos, cuya

relación entre la intensidad y la inducción magnética es no lineal. Bajo hipótesis fisicamente plausibles, se demostró que esta formulación está bien planteada y que el multiplicador de Lagrange es nuevamente nulo.

5. Tanto para el caso lineal como el no lineal, se propusó un esquema de elementos finitos basado en elementos de Nédélec para la variable principal y elementos finitos usuales para el multiplicador. En ambos casos se demostró que el esquema semidiscreto resultante está bien planteado y que genera aproximaciones de orden óptimo. Además, también para ambos casos, se obtuvo un esquema completamente discreto a través de una discretización en tiempo por el método de Euler implícito, que igualmente arrojó estimaciones optimas del error. Las aproximaciones obtenidas en el caso lineal proporcionan estimaciones para las magnitudes físicas más relevantes: las corrientes inducidas en el material conductor y la inducción magnética en el dominio computacional. En el caso no lineal, solo se consiguió estimar esta última magnitud.

6. Se demostró que la formulación del inciso **3)** puede extenderse para abarcar el problema evolutivo de corrientes inducidas en todo el espacio. La formulación que se obtiene, permite realizar un acoplamiento FEM-BEM para el problema usando las mismas variables del caso acotado y un potencial escalar en la frontera. Se demuestra que la formulación obtenida es un problema bien planteado. La elección del dominio computacional acotado simplemente conexo con frontera conexa, permite definir esquemas semidiscretos y completamente discretos, usando elementos finitos usuales para aproximar la variable de la frontera y aproximando las otras dos variables de la misma forma que en el caso acotado. Se dedujeron estimaciones de error similares a las del caso acotado, incluyendo estimaciones del error de la variable en la frontera.

6.2 Trabajo futuro

1. Extender el análisis de la formulación en potenciales (Capítulo 2) al problema evolutivo.

2. En colaboración con Edwin Behrens (U.C.S.C., Chile), se está implementando el método estudiado en los Capítulos 3-5 de esta tesis, con el propósito de desarrollar una experimentación numérica que permita analizar su eficiencia.

3. Extender los resultados de la formulación FEM-BEM estudiada en el capítulo 5 al problema no lineal magnético.

4. Estudiar la convergencia de la aproximación de las corrientes inducidas en el caso no lineal magnético.

Bibliography

[1] R. Acevedo, S. Meddahi, and R. Rodríguez, *An* $\boldsymbol{E}$*-based mixed formulation for a time-dependent eddy current problem*, Preprint 2008-03, Departamento de Ingeniería Matemática, Universidad de Concepción.

[2] R. Acevedo and R. Rodríguez, *Analysis of the* $\boldsymbol{A}$*,* $V - \boldsymbol{A} - \psi$ *potential formulation for the eddy current problem in a bounded domain*, Electron. Trans. Numer. Anal., 26 (2007), pp. 270–284.

[3] R. Albanese and G. Rubinacci, *Formulation of the eddy current problem*, IEEE proceedings, A 137 (1990), pp. 16–22.

[4] D. Albertz and G. Henneberger, *Calculation of 3D eddy current fields using both electric and magnetic vector potential in conducting regions*, IEEE Trans. Magn., 34 (1998), pp. 2644–2647.

[5] D. Albertz and G. Henneberger, *On the use of the new based* $\boldsymbol{A} - \boldsymbol{A}, \boldsymbol{T}$ *formulation for the calculation of time-harmonic stationary and transient eddy current field problems*, IEEE Trans. Magn., 36 (2000), pp. 818–822.

[6] E. Alexandre and J.-L. Guermond, *Theory and practice of finite elements*, Springer, New York, 2004.

[7] A. Alonso Rodríguez, P. Fernandez, and A. Valli, *Weak and strong formulations for the time-harmonic eddy-current problem in general multi-connected domains*, European J. Appl. Math., 14 (2003), pp. 387–406.

[8] A. Alonso Rodríguez, R. Hiptmair, and A. Valli, *Mixed finite element approximation of eddy current problems*, IMA J. Numer. Anal., 24 (2004), pp. 255–271.

[9] A. Alonso-Rodríguez, R. Hiptmair, and A. Valli, *Hybrid formulations of eddy current problems*, Numer. Methods PDEs, 21 (2005), pp. 742–763.

[10] A. Alonso Rodríguez and A. Valli, *A domain decomposition approach for heterogeneous time-harmonic Maxwell equations*, Comput. Methods Appl. Mech. Engrg., 143 (1997), pp. 97–112.

[11] A. Alonso Rodríguez and A. Valli, *An optimal domain decomposition preconditioner for low-frequency time-harmonic Maxwell equations*, Math. Comp., 68 (1999), pp. 607–631.

[12] A. Alonso Rodríguez and A. Valli, *A unified FEM-BEM approach for electromagnetostatics and time-harmonic eddy-current problems*, Technical Report UTM 715, June 2007, Matematica, University of Trento.

[13] H. Ammari, A. Buffa and J.-C. Nédélec, *A justification of eddy currents model for the Maxwell equations*, SIAM J. Appl. Math., 60 (2000), pp. .1805–1823

[14] C. Amrouche, C. Bernardi, M. Dauge, and V. Girault, *Vector potentials in three-dimensional non-smooth domains*, Math. Meth. Appl. Sci., 21 (1998), pp. 823–864.

[15] F. Bachinger, U. Langer and J. Schöberl, *Numerical analysis of nonlinear multiharmonic eddy current problems*, Numer. Math., 100 (2005), pp. 593–616.

[16] A. Bermúdez, R. Rodríguez, and P. Salgado, *A finite element method with Lagrange multipliers for low-frequency harmonic Maxwell equations*, SIAM J. Numer. Anal., 40 (2002), pp. 1823–1849.

[17] A. Bermúdez, R. Rodríguez, and P. Salgado, *Numerical solution of eddy current problems in bounded domains using realistic boundary conditions*, Comput. Methods Appl. Mech. Engrg., 194 (2005), pp. 411–426.

[18] A. Bermúdez, R. Rodríguez, and P. Salgado, *Numerical analysis of an electric formulation of the eddy current problem*, Numer. Math., 102 (2005), pp. 181–201.

[19] A. Bermúdez, R. Rodríguez, and P. Salgado, *Numerical treatment of realistic boundary conditions for the eddy currents problem in an electrode via Lagrange multipliers*, Math. Comp., 74 (2005), pp. 123–151.

[20] A. Bermúdez, R. Rodríguez, and P. Salgado, *FEM for 3D eddy current problems in bounded domains subject to realistic boundary conditions. An application to metallurgical electrodes*, Arch. Comput. Methods Eng., 12 (2005), pp. 67–114.

[21] C. Bernardi and G. Raugel, *A conforming finite element method for the time-dependent Navier-Stokes equations*, SIAM J. Numer. Anal., 22 (1985), pp. 455–473.

[22] O. Bíró *Edge element formulations of eddy current problems*, Comput. Meth. Appl. Mech. Eng., 169 (1999), pp. 391–405

[23] O. Bíró and K. Preis, *On the use of the magnetic vector potential in the finite element analysis of the three-dimensional eddy currents*, IEEE Trans. Magn., 25 (1989), pp. 3145–3159.

[24] O. Bíró and A. Valli, *The Coulomb gauged vector potential formulation for the eddy-current problem in general geometry: well-posedness and numerical approximation*, Comput. Meth. Appl. Mech. Eng., 196 (2007), pp. 1890–1904.

[25] D. Boffi and L. Gastaldi, *Analysis of finite element approximation of evolution problems in mixed form*, SIAM J. Numer. Anal., 42 (2004), pp. 1502–1526.

[26] A. Bossavit, *The computation of eddy-currents, in dimension* 3*, by using mixed finite elements and boundary elements in association*, Math. Comput. Modelling, 15 (1991), pp. 33–42.

[27] A. Bossavit, *Computational Electromagnetism*, Academic Press Inc., San Diego, CA, 1998.

[28] A. Bossavit, *"Hybrid" electric-magnetic methods in eddy-current problems*, Comput. Methods Appl. Mech. Engrg., 178 (1999), pp. 383–391.

[29] A. Bossavit and Vérité, *A mixed FEM/BIEM method to solve eddy-current problems*, IEEE Trans. Magn., 18 (1982), pp. 431–435.

[30] A. Bossavit and Vérité, *The TRIFOU code: Solving the 3-D eddy current by using* **H** *as state variable*, IEEE Trans. Magn., 19 (1983), pp. 2465–2470.

[31] A. Buffa and P. Ciarlet Jr., *On traces for functional spaces related to Maxwell equations. I. An integration by parts formula in Lipschitz polyhedra*, Math. Methods Appl. Sci., 24 (2001), pp. 9–30.

[32] A. Buffa and P. Ciarlet Jr., *On traces for functional spaces related to Maxwell equations. II. Hodge decompositions on the boundary of Lipschitz polyhedra and applications*, Math. Methods Appl. Sci., 24 (2001), pp. 31–48.

[33] A. Buffa, *Hodge decompositions on the boundary of nonsmooth domains: the multi-connected case*, Math. Models Methods Appl. Sci., 11 (2001), pp. 1491–1503.

[34] A. Buffa, M. Costabel, and D. Sheen, *On traces for* $\mathbf{H}(\mathbf{curl};\Omega)$ *in Lipschitz domains*, J. Math. Anal. Appl., 276 (2002), pp. 845–867.

[35] P. Ciarlet, *The Finite Element Method for Elliptic Problems*, SIAM, 2002.

[36] D. Colton and R. Kress, *Inverse acoustic and electromagnetic scattering theory*, Springer, Berlin, 1998.

[37] M. Costabel, *A symmetric method for the coupling of finite elements and boundary elements*, in: The mathematics of finite elements and applications VI, Academic Press, London, 1988, pp. 281–288.

[38] M. Costabel, *A coercive bilinear form for Maxwell's equations*, J. Math. Anal. Appl., 157 (1991), pp. 527–541.

[39] C. Daveau and J. Laminie, *Mixed and hybrid formulations for the three-dimensional magnetostatic problem*, Numer. Methods Partial Differential Equations, 18 (2002), pp. 85–104.

[40] M. Dauge, *Elliptic Boundary Value Problems on Corner Domains*, Lecture Notes in Math., 1341, Springer, Berlin, 1988.

[41] P. Fernandez and G. Gilardi, *Magnetostatic and electrostatic problems in inhomogeneous anisotropic media with irregular boundary conditions*, Math. Model. Meth. Appl. Sci., 7 (1997), pp. 957–991.

[42] P. Fernandez and A. Valli, *Lorenz-gauged vector potential formulations for the time-harmonic eddy-current problem with L^∞-regularity of material properties*, Math. Model. Meth. Appl. Sci., 31 (2008), pp. 71–98.

[43] V. Girault and P. A. Raviart, *Finite Element Methods for Navier Stokes Equations*, Springer, New York, 1986.

[44] B. Heise, *Analysis of a fully discrete finite element method for a nonlinear magnetic field problem*, SIAM J. Numer. Anal., 31 (1994), pp. 745–759.

[45] R. Hiptmair, *Finite elements in computational electromagnetism*, in Acta Numerica, 11, 2002, pp. 237–339.

[46] R. Hiptmair, *Symmetric coupling for eddy current problems*, SIAM J. Numer. Anal., 40 (2002), pp. 41–65.

[47] R. Hiptmair, *Boundary element methods for eddy current computation*, in Computational Electromagnetics, C. Carstensen, et al. eds., Lecture Notes in Computational Science and Engineering, 28, Springer, Berlin, 2003, pp. 103–126.

[48] C. T. A. Johnk, *Engineering electromagnetic fields and waves*, John Wiley & Sons, New York, 1988.

[49] C. Johnson and V. Thomée, *Error estimates for some mixed finite element methods for parabolic type problems*, RAIRO Anal. Numér., 15 (1981), pp. 41–78.

[50] M. Kuhn and O. Steinbach, *Symmetric coupling of finite and boundary elements for exterior magnetic field problems*, Math. Methods Appl. Sci., 25 (2002), pp. 357–371.

[51] N. A. Golias, C. S. Anntonopoulus, T. D. Tsiboukis, and E. E. Kriezis, *3D eddy current computation with edge elements in terms of the electric intensity*, COMPEL, 17 (1998), pp. 667–673.

[52] P. J. Leonard and D. Rodger, *Finite element scheme for transient 3D eddy currents*, IEEE Trans. Magn., 24 (1988), pp. 90-93.

[53] J. Laminie and S. M. Mefire, *Three-dimensional computation of a magnetic field by mixed finite elements and boundary elements*, Appl. Numer. Math., 35 (2000), pp. 221–244.

[54] I. D. Mayergoyz, M. V. K. Chari, and A. Konrad, *Boundary Galerkin's method for three-dimensional finite element electromagnetic field computation*, IEEE Trans. Magn., 19 (1983), pp. 2333–2336.

[55] S. Meddahi and V. Selgas, *A mixed-FEM and BEM coupling for a three-dimensional eddy current problem*, M2AN Math. Model. Numer. Anal., 37 (2003), pp. 291–318.

[56] S. Meddahi and V. Selgas, *An* **H**-*based FEM-BEM formulation for a time dependent eddy current problem*, Appl. Numer. Math., 58 (2008), pp. 1061–1083.

[57] P. Monk, *Finite Element Methods for Maxwell's Equations*, Oxford Clarendon Press, 2003.

[58] T. Morisue, *Magnetic vector potential and electric scalar potential in three-dimensional eddy current problem*, IEEE Trans. Magn., 18 (1982), pp. 531–535.

[59] T. Morisue, *3D-eddy current calculation using the magnetic vector potential*, IEEE Trans. Magn., 24 (1988), pp. 106–109.

[60] C. Müller, *Foundation of the Mathematical Theory of Electromagnetic Waves*, Springer-Verlag, Berlin, 1969.

[61] J.-C. Nédélec, *Mixed finite elements in* $\mathbf{R}^3$, Numer. Math., 35 (1980), pp. 315–341.

[62] A. Quarteroni and A. Valli, *Numerical Approximation of Partial Differential Equations*, Springer-Verlag, Berlin, 1994.

[63] A. B. J. Reece and T. W. Preston , *Finite Element Methods in Electrical Power Engineering*, Oxford University Press, New York, 2000.

[64] M. Reissel, *On a transmission boundary value problem for the time-harmonic Maxwell equations without displacement currents*, SIAM J. Math. Anal., 24 (1993), pp. 1440–1457.

[65] S. Reitzinger, B. Kaltenbacher and M. Kaltenbacher, *A note on the approximation of B-H curves for nonlinear magnetic field computations*, Johannes Keppler Universität Linz, SFB "Numerical and Symbolic Scientific Computing", SFB-Report No 02-01, 2002.

[66] D. Rodger and J. F. Eastham, *A formulation for low frequency eddy current solution*, IEEE Trans. Magn., MAG-19 (1983), pp. 2443–2446.

[67] P. P. Silvester and R. L. Ferrari, *Finite Elements for Electrical Engineers*, Cambridge University Press, Cambridge, 1996.

[68] J. K. Sykulski, *Computational Magnetics*, London, Chapman & Hall, 1995.

[69] H. T. Tu, K. R. Shao and K. D. Zhou, *H method for solving 3D eddy current problems*, IEEE Trans. Magn., 31 (1995), pp. 3518–3520.

[70] J. P. Webb and B. Forghani, *A escalar-vector method for 3D eddy current problems using edge elements*, IEEE Trans. Magn., 26 (1990), pp. 2367–2369.

[71] J. P. Webb and B. Forghani, *The low-frequency perfomance of $\boldsymbol{H}-\phi$ and $\boldsymbol{T}-\Omega$ methods using edge elements for 3D eddy current problems*, IEEE Trans. Magn., 29 (1993), pp. 2461–2463.

[72] J. P. Webb, B. Forghani and D. A. Lowther, *An approach to the solution of three-dimensional voltage driven and multiply connected eddy current problems*, IEEE Trans. Magn., 28 (1992), pp. 1193–1196.

[73] E. Zeidler, *Nonlinear Functional Analysis and its Applications. II/A*, Springer-Verlag, New York, 1990.

[74] E. Zeidler, *Nonlinear Functional Analysis and its Applications. II/B*, Springer-Verlag, New York, 1990.

[75] W. Zheng, Z. Chen, and L. Wang, *An adaptive finite element method for the H-ψ formulation of time-dependent eddy current problems*, Numer. Math., 103 (2006), pp. 667–689.