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The Cauchy problem for the Benney-Luke equation and generalized Benney-Luke equation

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Rio de Janeiro
August, 2008

To Maria Mercedes, my daughter

Acknowledgments

I thank to God for his blessings.

I would like to thank to my thesis advisor, Professor Felipe Linares for proposing me the problem studied in this investigation. I am grateful for his patience, dedication and support during my stay at IMPA. His excellent lectures in Nonlinear Dispersive Equations and Harmonic Analysis enriched my work.

I further express my gratefulness to Professors Gustavo Ponce, Rafael Iorio, Marciam Scialom and André Nachbin, who kindly accepted to participate in my thesis committee.

I thank to my husband Enrique and my family in Colombia for their patience, affection and support. They always encouraged me in difficult moments.

I would like to thank my friends for their affection and support.

Finally I thank to the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), CAPES/CNPq-IEL Nacional-Brasil and Universidad del Cauca Colombia for financing my doctoral studies in Brazil.

Abstract.

We examine the question of the minimal Sobolev regularity required to construct local solutions to the Cauchy problem for the isotropic Benney-Luke (BL), isotropic p-generalized Benney-Luke (p-gBL) and generalized Benney-Luke (gBL) equations. The main results in this work regards the global well-posedness of the initial value problem (IVP) associated to the (BL), the (p-gBL) and (gBL) equations in the energy space, $\dot{H}^2(\mathbb{R}^2)\cap\dot{H}^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)$, the local well-posedness of IVP associated to the (gBL) in $H^s(\mathbb{R}^2)\times H^{s-1}(\mathbb{R}^2)$ for $9/5 < s \leq 2$. The IVP associated to the (BL) is locally well-posed in $H^s(\mathbb{R}^3)\times H^{s-1}(\mathbb{R}^3)$ for $2 < s \leq 5/2$. We also study the Cauchy problem in the periodic case for the isotropic Benney-Luke and generalized Benney-Luke equations. In this case we have local well-posedness in $H^s(\mathbb{T}\times\mathbb{R})\times H^{s-1}(\mathbb{T}\times\mathbb{R})$ and $H^s(\mathbb{T}^2)\times H^{s-1}(\mathbb{T}^2)$ for $2 < s \leq 3$.

Contents

Notation

$\mathbb{R}$ Real numbers

$\partial_x^k u$ or $u_{x...x}$ Partial derivative of u in the variable x of order k

$B(x,r)$ means the open ball with center x and radio r.

If ξ is a vector in $\mathbb{R}^n$ then $|\xi|^2 = \xi_1^2 + ... + \xi_n^2$

$\|f\|_{L^p(X,\mu)} := \|f\|_p = (\int_X |f|^p d\mu)^{\frac{1}{p}}$, $1 \leq p < \infty$

$H_2^s(\mathbb{R}^n) := H^s$ Fractional Sobolev spaces of order s real, i.e.,

$H^s(\mathbb{R}^n) := \{f \in S'(\mathbb{R}^n) \ : \ (1+|\xi|^2)^{\frac{s}{2}}\widehat{f}(\xi) \in L^2(\mathbb{R}^n)\}$

$\|f\|_{H^s} := \|(1+|\xi|^2)^{s/2}\widehat{f}(\xi)\|_{L^2}$

$C([0,T]:X)$ Continuous functions from $[0,T]$ into X

$\hat{f}(\xi) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx$ Fourier Transform of f

$\check{f}(x) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)d\xi$ Inverse Fourier Transform of f

$S(\mathbb{R}^n)$ Schwartz space on $\mathbb{R}^n$

$J^s = (1-\Delta)^{s/2}$ denotes the Bessel potential of order $-s$

$|D|^s = (-\Delta)^{s/2}$ Riesz potential of order $-s$

$H_q^s(\mathbb{R}^n) := J^{-s}L^q(\mathbb{R}^n)$. When q=2 we will write H^s instead of H_2^s

$\|f\|_{H_q^s} := \|J^s \cdot\|_q$.

$\dot{H}_q^s(\mathbb{R}^n) := |D|^{-s}L^q(\mathbb{R}^n)$. $\dot{H}^s = |D|^{-s}L^2$

$f(x) \lesssim g(x)$ when exists a constant $C > 0$ such that $f(x) \leq Cg(x)$

Introduction.

An intermediate model for the evolution of weakly nonlinear, long water waves of small amplitude is given by the following equation

$$\Phi_{tt} - \Delta\Phi + \mu(a\Delta^2\Phi - b\Delta\Phi_{tt}) + \epsilon(\Phi_t\Delta\Phi + 2\nabla\Phi\cdot\nabla\Phi_t) = 0, \qquad (1)$$

where $\Phi(t, \mathbf{x})$ is a real valued function, $(t, \mathbf{x}) \in \mathbb{R}_+ \times \mathbb{R}^2$, $\mathbb{R}_+ = [0, \infty)$, a, b, μ, and ϵ are positive real constants and ∇ and Δ are the two-dimensional gradient and Laplacian, respectively.

In the equation (1), so-called isotropic Benney-Luke equation (BL), Φ is the velocity potential on the domain. After rescaling the variables, we can suppose that the constants a and b are positive and such that $a - b = \alpha - \frac{1}{3} \neq 0$, where α is the Bond number, ϵ (nonlinearity coefficient) is the amplitude parameter and $\mu = (h_0/L)^2$ is the long-wave parameter (dispersion coefficient), where h_0 is the equilibrium depth and L is the length scale. This equation was first derived by Benney and Luke (see [2]) when $a = 1/6$ and $b = 1/2$ with no surface tension ($\alpha = 0$).

Pego and Quintero [27] showed that the isotropic Benney-Luke (BL) equation reduces formally to the Kadomtsev-Petviashvili (KP-I or KP-II) equation after a suitable re-normalization. Indeed, putting $2\tau = \epsilon t$, $X = x - t$, $Y = \sqrt{\epsilon}y$ and $\Phi(t, x, y) = f(\tau, X, Y)$, neglecting $O(\epsilon)$ terms we find that

$\eta = f_X$ satisfies

$$(\eta_\tau - (\alpha - \frac{1}{3})\eta_{XXX} + 3\eta\eta_X)_X + \eta_{YY} = 0. \tag{2}$$

We recall that if $\alpha > 1/3$ this equation is KP-I, if $\alpha < 1/3$ it is KP-II and, if we suppose that f does not depend on the Y variable we obtain the Korteweg de-Vries (KdV) equation. They also found traveling-wave solutions of (1), i.e., solutions of the form $\Phi(t, x, y) = \frac{\sqrt{\mu}}{\epsilon} v(\frac{x-ct}{\sqrt{\mu}}, \frac{y}{\sqrt{\mu}})$ and showed that if the wave speed c satisfies $c^2 < \min(1, a/b)$ then there exists a nontrivial finite-energy solution v, where the energy associated to v is given by

$$E(v) = \frac{1}{2}\int_{\mathbb{R}^2} \{(1+c^2)v_x^2 + v_y^2 + (a+bc^2)v_{xx}^2 + (2a+bc^2)v_{xy}^2 + av_{yy}^2\}dxdy. \tag{3}$$

Quintero in [30] proved that the solitary waves are orbitally stable if the wave speed c is near 0 or 1. He also showed in [29] the existence and analyticity of the lump solution for isotropic p-generalized Benney-Luke equation

$$\Phi_{tt} - \Delta\Phi + \mu(a\Delta^2\Phi - b\Delta\Phi_{tt}) + \epsilon(\Phi_t\Delta_p\Phi + 2\nabla^p\Phi \cdot \nabla\Phi_t) = 0, \tag{4}$$

where ∇^p and Δ_p are given by

$$\nabla^p\Phi = ((\partial_x\Phi)^p, (\partial_y\Phi)^p) \tag{5}$$

$$\Delta_p\Phi = \nabla \cdot (\nabla^p\Phi) = \partial_x(\partial_x\Phi)^p + \partial_y(\partial_y\Phi)^p. \tag{6}$$

From now on we will call the equations (1) and (4) as (BL) and (p-gBL) equations, respectively . The family of isotropic Benney-Luke equations includes the effect of surface tension and a variety of equivalent forms of dispersion. Let us remark that the model (1) does not hold for $a = b$ ($\alpha =$

1/3). Paumond in [25], derived an equation that is still valid when we suppose that α is equal or close to 1/3. More precisely,

$$
\begin{aligned}
&\Phi_{tt} - \Delta\Phi + \sqrt{\epsilon}(a\Delta^2\Phi - b\Delta\Phi_{tt}) + \epsilon(B\Delta^2\Phi_{tt} - A\Delta^3\Phi) \\
&\qquad + \epsilon(\Phi_t\Delta\Phi + 2\nabla\Phi\cdot\nabla\Phi_t) = 0,
\end{aligned}
\tag{7}
$$

where $\epsilon = \mu^2$ and the parameters A, B are linked. In [24], it was rigorously shown that the $L^2(\mathbb{R}^2)$-norm of the difference between the amplitude of the wave given by equation (2) and the one given by isotropic Benney-Luke (BL) equation is of order $O(\epsilon^{3/4})$ during a growing with ϵ time. Paumond in [24] also studied the Cauchy problem

$$
\begin{cases}
\Phi_{tt} - \Delta\Phi + \mu(a\Delta^2\Phi - b\Delta\Phi_{tt}) + \epsilon(\Phi_t\Delta\Phi + 2\nabla\Phi\cdot\nabla\Phi_t) = 0 \\
\Phi(0,\mathbf{x}) = \Phi_0(\mathbf{x}), \quad \Phi_t(0,\mathbf{x}) = \Phi_1(\mathbf{x}),
\end{cases}
\tag{8}
$$

and proved that it is globally well-posedness for initial data in $H^s(\mathbb{R}^2) \times H^{s-1}(\mathbb{R}^2)$, s integer and $s \geq 2$.

It is known that most water wave models are equipped with a Hamiltonian structure. Moreover, global results concerning existence and uniqueness of solutions for the associated Cauchy problems follow by the existence of some conserved quantities. It is also known that the natural space to consider the well-posedness of such initial value problems is dictated by the well definition of either the Hamiltonian or the energy. For the problem (8), the Hamiltonian and the energy are well defined if $\Phi \in \dot{H}^2(\mathbb{R}^2) \cap \dot{H}^1(\mathbb{R}^2)$ and $\Phi_t \in H^1(\mathbb{R}^2)$. Our main purpose is to prove global well-posedness in the energy space. In this work we also study the local regularity and the well-posedness of the so-called generalized Benney-Luke equation

$$
\begin{cases}
u_{tt} - \Delta u_{tt} + \Delta^2 u - m_0\Delta u + \alpha_0(u_t\Delta u + 2\nabla u\cdot\nabla u_t) + \beta\nabla\cdot(|\nabla u|^m\nabla u) = 0 \\
u(0,\mathbf{x}) = u_0(\mathbf{x}), \quad u_t(0,\mathbf{x}) = u_1(\mathbf{x}),
\end{cases}
\tag{9}
$$

where $u(t, \mathbf{x})$ is a real valued function, $(t, \mathbf{x}) \in \mathbb{R}_+ \times \mathbb{R}^2$, $\mathbb{R}_+ = [0, \infty)$, m_0 and m are positive real constants, β and α_0 constants. The equation (9) with $m = 2$ is a model to describe dispersive and weakly nonlinear long water waves with small amplitude. If we omit the last term on the left side of the equation (9), one can obtain a rescaled version of the isotropic Benney-Luke equation (1).

The notion of local well-posedness to be used here is in the sense of Kato, that is, we will say that an *initial value problem* (IVP) is *locally well-posed* in some functional space X, if for all initial datum ϕ in X, there exists a time $T > 0$ and a unique solution u of the integral equation associated to the IVP (*existence and uniqueness*), such that $u \in C([0, T]; X)$ (*persistence*) and the flow map data-solution is (at least) continuous from a neighborhood of ϕ in X into $C([0, T]; X)$ (*continuous dependence*). If T can be taken arbitrarily large, we say the well-posedness is *global*.

It is important to remember that the solitary wave solution of the IVP associated to (BL) equations lies in the energy space and it is orbitally stable if the wave speed c is near 0 or 1, (see [30]).

We obtain the local well-posedness result for isotropic Benney-Luke equations using the fixed point argument and the generalized Strichartz inequalities for the wave equation. We showed a similar result for the isotropic p-generalized Benney-Luke equation (p-gBL) and generalized Benney-Luke (gBL) equation. We also prove that the lower bound for the Sobolev exponent can be reduced from 5/2 to 2 in three space dimensions using the Strichartz estimates and the ideas of Ponce and Sideris [28] for all equations here considered.

In our case if we define u such that

$$\Phi(t,\mathbf{x}) := \frac{\sqrt{\mu}}{\sqrt[p]{\epsilon}} u\left(\frac{t}{\sqrt{\mu}}, \frac{1}{\sqrt{\mu}}\mathbf{x}\right), \tag{10}$$

with Φ satisfying the isotropic Benney-Luke equation or the isotropic p-generalized Benney-Luke equation ($p=1$ or with $p>1$, respectively), then the initial value problem associated to isotropic p-generalized Benney-Luke equation (4) is equivalent to

$$\begin{cases} (1-b\Delta)(u_{tt}-c^2\Delta u) = (1-c^2)\Delta u - F_p(\partial_t u, \nabla^p u, \nabla\partial_t u) \\ u(0,\mathbf{x}) = u_0(\mathbf{x}) \quad u_t(0,\mathbf{x}) = u_1(\mathbf{x}), \end{cases} \tag{11}$$

where $c^2 = \dfrac{a}{b}$, $u_i(\mathbf{x}) = \dfrac{\sqrt[p]{\epsilon}}{\sqrt{\mu}}\Phi_i(\sqrt{\mu}\,\mathbf{x})$, $i=0,1$, $\mathbf{x}\in\mathbb{R}^2$ and

$$\begin{aligned} F_p(\partial_t u, \nabla^p u, \nabla\partial_t u) =& p\partial_t u(u_x)^{p-1}u_{xx} + p\partial_t u(u_y)^{p-1}u_{yy} \\ &+ 2\partial_t u_x(u_x)^p + 2\partial_t u_y(u_y)^p. \end{aligned} \tag{12}$$

We notice that the energy method and Sobolev inequalities yield local well-posedness results in $H^s(\mathbb{R}^n)\times H^{s-1}(\mathbb{R}^n)$ with $s>n/2+1$ for the IVP associated to the following systems of nonlinear wave equations

$$-u^I_{tt} + \Delta u^I = F(u, Du), \tag{13}$$

where the vector $u=(u^1,...,u^N)$ depends on the variables $t=x_0, x_1, ...x_n$, $Du=(\partial_\alpha u^I)$, $\alpha=0,1,...,n$, $I=1,...,N$.

We also observe that Klainerman and Machedon [17], proved that if the nonlinear terms $F(u,Du)$ in (13) have the form

$$F^I = \sum_{J,K}\Gamma^I_{J,K}(u)B^I_{J,K}(Du^J, Du^K)$$

where the $B^I_{J,K}$ are expressions of the form

$$Q_0(\phi,\psi) = -\partial_t\phi\partial_t\psi + \sum_{i=1}^n \partial_i\phi\partial_i\psi$$

or

$$Q_{\alpha\beta}(\phi,\psi)=\partial_\alpha\phi\partial_\beta\psi-\partial_\beta\phi\partial_\alpha\psi,\quad 0\le\alpha<\beta\le n,$$

called *null forms*, (see [16]) then the IVP associated to (13) is locally well-posed in $H^s(\mathbb{R}^3)\times H^{s-1}(\mathbb{R}^3)$ for $s=2$.

The nonlinear term

$$(1-b\Delta)^{-1}F_p(\partial_t u,\nabla^p u,\nabla\partial_t u)$$

in the equation of the IVP (11) does not satisfy a "null condition" (see [16], for a definition) but it is still possible to prove that the IVP (11) is locally well-posed in $H^s(\mathbb{R}^3)\times H^{s-1}(\mathbb{R}^3)$ for $2<s\le 5/2$, in $H^s(\mathbb{R}^2)\times H^{s-1}(\mathbb{R}^2)$ for $s=2$ and in $\dot{H}^2(\mathbb{R}^2)\cap\dot{H}^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)$. This is possible using the Strichartz estimates, the commutators Kato-Ponce type [14] and the ideas of Ponce-Sideris in [28, inequality (12)].

We will consider the Cauchy problem (11) instead of the initial value problem associated to (BL) and (p-gBL), notice that F_p is the nonlinear term of isotropic Benney-Luke equation when $p=1$.

This work is divided as follows: In the first chapter we give some preliminaries, including linear estimates and Strichartz inequalities, which are detailed in the appendix.

In the second part we will prove the results of local and global well-posedness of the isotropic Benney-Luke equation (BL) for initial data in $H^2(\mathbb{R}^2)\times H^1(\mathbb{R}^2)$ and certain local regularity of solutions, such as $\nabla u(t),u_t(t)\in L^\infty$ a.e. $t\in(0,T)$. The main result, in this chapter, is the global well-posedness in the energy space, $\dot{H}^2(\mathbb{R}^2)\cap\dot{H}^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)$. For the 3-dimensional case the results are local in $H^s(\mathbb{R}^3)\times H^{s-1}(\mathbb{R}^3)$ for $2<s\le 5/2$

and again some results of local regularity, as above mentioned $\nabla u(t), u_t(t) \in L^\infty$ a.e. $t \in (0,T)$.

In the third chapter we will study the initial valued problem associated to the so called isotropic p-generalized Benney-Luke equation (p-gBL). In particular we establish global well-posedness in the energy space.

Next, in the chapter 4 we turn out to generalized Benney-Luke equation (9) and we prove that the associated Cauchy problem is locally well-posed in $H^s(\mathbb{R}^2) \times H^{s-1}(\mathbb{R}^2)$ for $9/5 < s \leq 2$ including the case of physical interest ($m = 2$). We also obtain some regularity results. As a consequence of local results we can establish global well-posedness in the energy space, $\dot{H}^2(\mathbb{R}^2) \cap \dot{H}^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ for $\beta < 0$.

Finally in the chapter 5, we study local well-posedness of the generalized Benney-Luke equations in the periodic setting. We prove that the IVP associated is locally well-posed in $H^s(\mathbb{T}\times\mathbb{R})\times H^{s-1}(\mathbb{T}\times\mathbb{R})$ and $H^s(\mathbb{T}^2)\times H^{s-1}(\mathbb{T}^2)$ for $2 < s \leq 3$.

Chapter 1

Preliminaries

Notation

The notation to be used is mostly standard. For any $q \in [1,\infty]$, we denote by q' its conjugate exponent, i.e. $\frac{1}{q}+\frac{1}{q'}=1$. Let $L^q := L^q(\mathbb{R}^n)$ be the Lebesgue space, the norm on L^q is denoted by $\|\cdot\|_q$. The homogeneous spaces and the Sobolev spaces $\dot{H}^s_q(\mathbb{R}^n)$ and $H^s_q(\mathbb{R}^n)$, respectively, are defined by $(-\Delta)^{-s/2}L^q(\mathbb{R}^n)$ and $J^{-s}L^q(\mathbb{R}^n)$ with $J := (1-\Delta)^{1/2}$. We denote $\dot{H}^s_2(\mathbb{R}^n)$ and $H^s_2(\mathbb{R}^n)$ by $\dot{H}^s$ and H^s, respectively.

The norms on $\dot{H}^s_q(\mathbb{R}^n)$ and $H^s_q(\mathbb{R}^n)$ are denoted by $\|\cdot\|_{\dot{H}^s_q}$ and $\|\cdot\|_{H^s_q}$, respectively. We will use the Sobolev spaces $L^r_t\dot{H}^\rho_q(\mathbb{R}^n)$ and $L^r_T\dot{H}^\rho_q(\mathbb{R}^n)$ endowed with the norm

$$\|u\|_{L^r_t\dot{H}^\rho_q} = \left(\int_{\mathbb{R}} \|u(t)\|^r_{\dot{H}^\rho_q} dt\right)^{1/r}, \quad \|u\|_{L^r_T\dot{H}^\rho_q} = \left(\int_0^T \|u(t)\|^r_{\dot{H}^\rho_q} dt\right)^{1/r}.$$

Throughout this work $C \geqq 0$ (independent of the data of the problem) will stand for a constant that can change from line to line. For any positive numbers a and b, $a \lesssim b$ means that $a \leqslant Cb$ for some constant C greater than zero. We also denote $a \sim b$ when $a \lesssim b$ and $b \lesssim a$.

To show our results we will use some estimates for solutions of linear problem and conmutators estimates as the commutators of Kato-Ponce type [14].

1.1 Linear Estimates

The linear problem associated to (11) is

$$\begin{cases} u_{tt} - \Delta u + a\Delta^2 u - b\Delta u_{tt} = 0 \\ u(0,x) = f(x), \qquad u_t(0,x) = g(x). \end{cases} \tag{1.1}$$

Let $h(\xi) = \left(\dfrac{1+a|\xi|^2}{1+b|\xi|^2}\right)^{1/2}$,

$$(\widehat{W(t)g})(\xi) = (|\xi|h(\xi))^{-1} \sin\left(|\xi|h(\xi)t\right) \hat{g}(\xi)$$

and

$$(\widehat{\dot{W}(t)f})(\xi) = \cos\left(|\xi|h(\xi)t\right) \hat{f}(\xi).$$

Then a solution of

$$\begin{cases} u_{tt} - \Delta u + a\Delta^2 u - b\Delta u_{tt} = G(u) \\ u(0,\cdot) = f(\cdot), \quad u_t(0,\cdot) = g(\cdot) \end{cases} \tag{1.2}$$

when $a \neq b$ and f, g are smooth is given by

$$u(t) = \dot{W}(t)f + W(t)g + \int_0^t W(t-t')G(u)(t')dt'. \tag{1.3}$$

If $a = b$ and f, g are smooth the solution of (1.2) is

$$u(t) = \dot{\mathbf{K}}(t)f + \mathbf{K}(t)g + \int_0^t \mathbf{K}(t-t')(1-b\Delta)^{-1}G(u)(t')dt', \tag{1.4}$$

where $\{\mathbf{K}(t)\}_t$ is the classical wave semi-group,

$$(\widehat{K(t)g})(\xi) = |\xi|^{-1}\sin(|\xi|t)\hat{g}(\xi)$$

with

$$(\widehat{\dot{\mathbf{K}}(t)f})(\xi) = \cos(|\xi|t)\hat{f}(\xi),$$

and $(1 - b\Delta)^{-1}G(u)$ is defined via the Fourier transform as

$$((1 - b\Delta)^{-1}G(u))\widehat{\ }(\xi) = (1 + b|\xi|^2)^{-1}\widehat{G(u)}(\xi).$$

It is clear that $W(t)$ is bounded in $L^2(\mathbb{R}^n)$, for all $a, b > 0$, since

$$\begin{aligned} \|W(t)g\|_2 &= \|(\widehat{W(t)g})(\cdot)\|_2 \\ &\leq \|\,(|\cdot|h(\cdot))^{-1}\sin(|\cdot|h(\cdot)t)\,\|_\infty \|\hat{g}(\cdot)\|_2 \end{aligned} \tag{1.5}$$

and $\|h\|_\infty \leq \max\{1, \sqrt{a/b}\}$.

Then

$$\|W(t)g\|_2 \lesssim |t|\|g\|_2. \tag{1.6}$$

Moreover, for all $s \geq 0$

$$\begin{aligned} &\|W(t)g\|_{\dot{H}^s} \leq \max\{1, \sqrt{a/b}\}\|g\|_{\dot{H}^{s-1}} \\ &\|\dot{W}(t)f\|_{\dot{H}^s} \leq \|f\|_{\dot{H}^s}. \end{aligned} \tag{1.7}$$

Remark 1.1.1. *If we write equation (11) when $p = 1$ as*

$$\begin{cases} (1 - bc^{-2}\Delta)(u_{tt} - \Delta u) = c^{-2}(1 - c^2)\Delta u - c^{-2}\left(u_t\Delta u + 2\nabla u \cdot \nabla u_t\right) \\ u(0, \mathbf{x}) = u_0(\mathbf{x}), \quad u_t(0, \mathbf{x}) = u_1(\mathbf{x}), \end{cases} \tag{1.8}$$

we will see that it is sufficient to have the estimates (1.6) and (1.7) for $\mathbf{K}(t)$.

1.2 Strichartz Estimates for $\mathbf{K}(t)$ and $\dot{\mathbf{K}}(t)$

The Strichartz estimates constitute the main tool used to demostrate local regularity and establish well posedness in the energy space.

We are interested in the Cauchy problem for the wave equation

$$\begin{cases} u_{tt} - \Delta u = f \\ u(0, \mathbf{x}) = u_0(\mathbf{x}), \qquad u_t(0, \mathbf{x}) = u_1(\mathbf{x}), \end{cases} \tag{1.9}$$

and we denote the operators $\omega = (-\Delta)^{1/2}$, and $U(t) = exp(i\omega t)$, $\mathbf{K}(t) = (\omega)^{-1} \sin \omega t$ and $\dot{\mathbf{K}}(t) = \cos \omega t$. The Cauchy problem (1.9) is solved by $u = v + w$ where v is the solution of the homogeneous equation with the same data

$$\begin{cases} v_{tt} - \Delta v = 0 \\ v(0, \mathbf{x}) = u_0(\mathbf{x}), \qquad v_t(0, \mathbf{x}) = u_1(\mathbf{x}), \end{cases} \tag{1.10}$$

therefore

$$v(t) = \dot{\mathbf{K}}(t) u_0 + \mathbf{K}(t) u_1 \tag{1.11}$$

$$v_t(t) = \mathbf{K}(t) \Delta u_0 + \dot{\mathbf{K}}(t) u_1, \tag{1.12}$$

and w is the solution for the inhomogeneous equation with zero data,

$$\begin{cases} w_{tt} - \Delta w = f \\ w(0, \mathbf{x}) = 0, \qquad w_t(0, \mathbf{x}) = 0. \end{cases} \tag{1.13}$$

Let $L(t)$ be any of the operators $\omega^\lambda U(t)$, $\omega^\lambda \mathbf{K}(t)$ or $\omega^\lambda \dot{\mathbf{K}}(t)$ with $\lambda \in \mathbb{R}$ and χ be the characteristic function of $\mathbb{R}$ in time. We define $L_R(t) = \chi_+(t) L(t)$ and $L_A(t) = \chi_-(t) L(t)$ where R and A stand for retarded and advanced. Then the Cauchy problem (1.13) is solved por positive time by

$$w(t) = \int_0^t \mathbf{K}(t - t') f(t') dt' = (\mathbf{K}_R *_t \chi_+ f)(t). \tag{1.14}$$

$$w_t(t) = \int_0^t \dot{\mathbf{K}}(t - t') f(t') dt' = (\dot{\mathbf{K}}_R *_t \chi_+ f)(t). \tag{1.15}$$

Similar formulas with advanced operators solve the Cauchy problem (1.13) for negative times. We restrict our attention from now on to positive times.

The initial data (u_0, u_1) for the the problem (1.9) will be taken in the space

$$Y^\mu \equiv \dot{H}^\mu(\mathbb{R}^n) \times \dot{H}^{\mu-1}(\mathbb{R}^n) \tag{1.16}$$

with $n \geq 2$ and $\mu \in \mathbb{R}$.

Proposition 1.2.1. *If $n \geq 2$, $2 \leq r_1, r_2 \leq \infty$, $2 \leq q_1, q_2 < \infty$, $\rho_1, \rho_2, \mu \in \mathbb{R}$ satisfy*

$$0 \leq \frac{2}{r_i} \leq \min\left\{1, (n-1)\left(\frac{1}{2} - \frac{1}{q_i}\right)\right\} \quad i = 1, 2, \tag{1.17}$$

$$\left(\frac{2}{r_i}, (n-1)\left(\frac{1}{2} - \frac{1}{q_i}\right)\right) \neq (1, 1) \quad i = 1, 2 \tag{1.18}$$

$$\rho_1 + n\left(\frac{1}{2} - \frac{1}{q_1}\right) - \frac{1}{r_1} = \mu \tag{1.19}$$

$$\rho_1 + n\left(\frac{1}{2} - \frac{1}{q_1}\right) - \frac{1}{r_1} = 1 - \left(\rho_2 + n\left(\frac{1}{2} - \frac{1}{q_2}\right) - \frac{1}{r_2}\right), \tag{1.20}$$

then the generalized Strichartz estimates for $\mathbf{K}(t)$ and $\dot{\mathbf{K}}(t)$ are given by:

1. Let $(u_0, u_1) \in Y^\mu$ (see (1.16)). Then v define by (1.11) satisfies the estimates

$$\|v\|_{L_t^{r_1}\dot{H}_{q_1}^{\rho_1}} + \|\partial_t v\|_{L_t^{r_1}\dot{H}_{q_1}^{\rho_1-1}} \leq C\left(\|u_0\|_{\dot{H}^\mu} + \|u_1\|_{\dot{H}^{\mu-1}}\right), \tag{1.21}$$

2. For any interval $I = [0, T)$, $0 < T \leq \infty$, the function w defined by (1.14) satisfies the estimates

$$\|w\|_{L_T^{r_1}\dot{H}_{q_1}^{\rho_1}} + \|\partial_t w\|_{L_T^{r_1}\dot{H}_{q_1}^{\rho_1-1}} \leq C\|f\|_{L_T^{r_2'}\dot{H}_{q_2'}^{-\rho_2}}. \tag{1.22}$$

Proof. See [10] and the appendix. □

1.3 Nonlinear Estimates

Proposition 1.3.1. *If* $f, g \in \mathcal{S}(\mathbb{R}^n)$*,* $s \in \mathbb{Z}^+$*, then exists* $C = C_{n,s} > 0$ *such that*

$$\begin{aligned}\sum_{|\alpha|=s} \|[\partial_x^\alpha, f]g\|_2 =& \sum_{|\alpha|=s} \|\partial_x^\alpha(fg) - f\partial_x^\alpha g\|_2 \\ \leq& C(\|\nabla f\|_\infty \sum_{|\beta|=s-1} \|\partial_x^\beta g\|_2 + \|g\|_\infty \sum_{|\beta|=s} \|\partial_x^\beta f\|_2).\end{aligned} \tag{1.23}$$

Proof. It follows by the Leibniz rule and the Gagliardo-Nirenberg Inequality. □

Proposition 1.3.2. *[* **Commutators of Kato-Ponce type** *] If* $f, g \in \mathcal{S}(\mathbb{R}^n)$*,* $s \geq 1$*, then there exists* $C = C_{n,s} > 0$ *such that*

$$\|[J^s; f]g\|_2 \leq C(\|\nabla f\|_\infty \|J^{s-1}g\|_2 + \|g\|_\infty \|J^s f\|_2). \tag{1.24}$$

Proof. See [14]. □

Lemma 1.3.3. *If* $f \in \mathcal{S}(\mathbb{R}^2)$*,* $2 < q < \infty$*,* $\sigma = \frac{9}{8} + \frac{3}{4q}$ *and* $0 < s_0 = 1 - \frac{2}{q} < 1$*, then*

$$\|f\|_\infty \leq C\left(\|f\|_{\dot{H}^{s_0}} + \|f\|_{\dot{H}_q^{2\sigma-2}}\right). \tag{1.25}$$

Proof. By Sobolev's inequality we have

$$\begin{aligned}\|f\|_\infty \leq& C\|(1-\Delta)^{1/q+}f\|_q \\ \leq& C\|f\|_q + C\|(-\Delta)^{1/q+}f\|_q \\ \leq& C\|(-\Delta)^{s_0/2}f\|_2 + C\|(-\Delta)^{\sigma-1}f\|_q\end{aligned} \tag{1.26}$$

since our assumptions imply that $1/q < \sigma - 1$ we have the result. □

Lemma 1.3.4. *If* $f, g \in \mathcal{S}(\mathbb{R}^2)$*,* $2 < q < \infty$*,* $s_0 = 1 - \frac{2}{q}$ *and* $\sigma = \frac{9}{8} + \frac{3}{4q}$ *then*

$$\begin{aligned}\|[\Delta, f]g\|_2 \leq& C\{\|\nabla f\|_{\dot{H}^{s_0}} + \|(-\Delta)^{\sigma-1/2}f\|_q\}\|g\|_{\dot{H}^1} \\ &+ C\{\|g\|_{\dot{H}^{s_0}} + \|(-\Delta)^{\sigma-1}g\|_q\}\|f\|_{\dot{H}^2}.\end{aligned} \tag{1.27}$$

Proof. By Proposition 1.3.1 with $s = 2$, we have

$$\|[\Delta, f]g\|_2 \leq C\left(\|\nabla f\|_\infty \|g\|_{\dot{H}^1} + \|g\|_\infty \|f\|_{\dot{H}^2}\right). \tag{1.28}$$

An application of Lemma 1.3.3 yields the result. □

Lemma 1.3.5. *If $f, g \in \mathcal{S}(\mathbb{R}^2)$. Fix $7/4 < s < 2$ and let $\frac{2}{4s-7} < q < \infty$ and $\sigma = \sigma(s) = \frac{s}{2} + \frac{1}{8} + \frac{3}{4q}$ and $0 < s_0 = 1 - \frac{2}{q} < 1$, then*

$$\|f\|_{L^\infty(\mathbb{R}^2)} \leq C(\|f\|_{\dot{H}^{s_0}(\mathbb{R}^2)} + \|f\|_{\dot{H}^{2\sigma-2}_q(\mathbb{R}^2)}).$$

Proof. By the Sobolev inequality we have

$$\begin{aligned}\|f\|_\infty &\leq \|(1-\Delta)^{1/q+} f\|_q \\ &\leq \|f\|_q + \|(-\Delta)^{1/q+} f\|_q \\ &\leq \|(-\Delta)^{s_0/2} f\|_2 + \|(-\Delta)^{\sigma-1} f\|_q.\end{aligned} \tag{1.29}$$

Since our assumptions imply that $1/q < \sigma - 1$ we have the result. □

Lemma 1.3.6. *If $f, g \in \mathcal{S}(\mathbb{R}^3)$ and $2 < s \leq 5/2$, let $1/(s-2) < q < \infty$ and $\sigma = \sigma(s) = \dfrac{s}{2} + \dfrac{1}{q}$ then*

$$\begin{aligned}\|g\|_{L^\infty(\mathbb{R}^3)} &\leq C\left(\|g\|_{H^{s-1}(\mathbb{R}^3)} + \|g\|_{\dot{H}^{2\sigma-2}_q(\mathbb{R}^3)}\right), \\ \|\nabla f\|_{L^\infty(\mathbb{R}^3)} &\leq C\left(\|f\|_{H^s(\mathbb{R}^3)} + \|f\|_{\dot{H}^{2\sigma-1}_q(\mathbb{R}^3)}\right).\end{aligned} \tag{1.30}$$

Proof. The proof is essentially that given in Lemma 1.3.3, see [28, inequality (12)]. □

Lemma 1.3.7. *If $f, g \in \mathcal{S}(\mathbb{R}^3)$ and $2 < s \leq 5/2$, let $1/(s-2) < q < \infty$ and $\sigma = \sigma(s) = \dfrac{s}{2} + \dfrac{1}{q}$, then*

$$\begin{aligned}\|[\Delta, f]g\|_{L^2(\mathbb{R}^3)} &\leq C\{\|f\|_{H^s(\mathbb{R}^3)} + \|(-\Delta)^{\sigma-1/2} f\|_{L^q(\mathbb{R}^3)}\}\|g\|_{\dot{H}^1(\mathbb{R}^3)} \\ &\quad + C\{\|g\|_{H^{s-1}(\mathbb{R}^3)} + \|(-\Delta)^{\sigma-1} g\|_{L^q(\mathbb{R}^3)}\}\|f\|_{\dot{H}^2(\mathbb{R}^3)}.\end{aligned} \tag{1.31}$$

Proof. By Proposition 1.3.1 with $s = 2$, we have

$$\|[\Delta, f]g\|_2 \leq C\left(\|\nabla f\|_\infty \|g\|_{\dot{H}^1} + \|g\|_\infty \|f\|_{\dot{H}^2}\right). \tag{1.32}$$

An application of Lemma 1.3.6 yields the result. □

Lemma 1.3.8. *For* $0 < s < 1$ *we have*

$$\|w\|_{\dot{H}^{s+1}} \lesssim \|w\|_{\dot{H}^1}^{1-s} \|w\|_{\dot{H}^2}^{s}. \tag{1.33}$$

Proof. It follow by Hölder's inequality with $p = \dfrac{1}{s}$ and $q = \dfrac{1}{1-s}$, (see [3]). □

Lemma 1.3.9. *If* $w \in \mathcal{S}(\mathbb{R}^2)$, $2 < q < \infty$, $\sigma = \dfrac{9}{8} + \dfrac{3}{4q}$ *and* $0 < s_0 = 1 - \dfrac{2}{q} < 1$, *then*

$$\|\nabla w\|_{L^\infty(\mathbb{R}^2)} \lesssim \|w\|_{\dot{H}^1(\mathbb{R}^2)}^{1-s_0} \|w\|_{\dot{H}^2(\mathbb{R}^2)}^{s_0} + \|w\|_{\dot{H}_q^{2\sigma-1}(\mathbb{R}^2)}. \tag{1.34}$$

Proof. It follow by Lemma 1.3.3 and the interpolation result (1.33). □

Chapter 2

Local and global well-posedness of isotropic Benney-Luke equation and local regularity of solutions

In this chapter we will study the well-posedness of the IVP associated to isotropic Benney-Luke equation in the following equivalent form

$$\begin{cases} (1-b\Delta)(u_{tt}-c^2\Delta u) = (1-c^2)\Delta u - F(u_t, \nabla u, \nabla u_t) \\ u(0,\mathbf{x}) = u_0(\mathbf{x}) \quad u_t(0,\mathbf{x}) = u_1(\mathbf{x}) \end{cases} \tag{2.1}$$

where

$$\Phi(t,\mathbf{x}) := \frac{\sqrt{\mu}}{\epsilon} u\left(\frac{t}{\sqrt{\mu}}, \frac{1}{\sqrt{\mu}}\mathbf{x}\right),$$

$F(u_t, \nabla u, \nabla u_t) = u_t \Delta u + 2\nabla u_t \cdot \nabla u$, $c^2 = \dfrac{a}{b}$, $u_i(\mathbf{x}) = \dfrac{\epsilon}{\sqrt{\mu}}\Phi_i(\sqrt{\mu}\,\mathbf{x})$, $i = 0, 1$ and $\mathbf{x} \in \mathbb{R}^2$.

2.1 Main results

We show that the local solution of the IVP (2.1) possesses certain local regularity, like for example $\nabla\Phi(t), \Phi_t(t) \in L^\infty(\mathbb{R}^2)$ a.e. in $t \in (0, T)$.

The next result recovers the one obtained by Paumond in [24], but by using the Strichartz inequalities for the wave equation.

Theorem 2.1.1 (Local well-posedness and local regularity)**.** *Assume that* $u_0 \in H^2(\mathbb{R}^2)$ *and* $u_1 \in H^1(\mathbb{R}^2)$. *Then there exists* $T = T(\|u_0\|_{H^2}, \|u_1\|_{H^1}) > 0$ *such that (2.1) has a unique solution* u *satisfying*

$$u \in C(0,T;H^2(\mathbb{R}^2)), \quad u_t \in C(0,T;H^1(\mathbb{R}^2)).$$

And u *has the following local regularity*

$$\left(\int_0^T \|(-\Delta)^{\sigma-1/2}u(t,\cdot)\|_q^r dt\right)^{1/r} < \infty, \quad \left(\int_0^T \|(-\Delta)^{\sigma-1}u_t(t,\cdot)\|_q^r dt\right)^{1/r} < \infty,$$

$$\int_0^T \|(\nabla u, u_t)(t)\|_\infty^4 dt < \infty,$$

with $r = \dfrac{4q}{q-2}$, $\sigma = \dfrac{9}{8} + \dfrac{3}{4q}$ *and* $2 < q < \infty$.

Moreover, for all $0 < T' < T$ *there exists a neighborhood* V *of* $(u_0, u_1) \in H^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ *such that the map data solution*

$$\begin{array}{ccc} V & \to & C(0,T';H^2(\mathbb{R}^2)) \cap L^r(0,T';\dot{H}_q^{2\sigma-1}(\mathbb{R}^2)) \\ (\tilde{u}_0, \tilde{u}_1) & \to & \tilde{u}(t) \end{array}$$

is Lipschitz.

The main result for the (BL) equation is in $\dot{H}^1(\mathbb{R}^2) \cap \dot{H}^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$.

Theorem 2.1.2 (Local well-posedness in the energy space)**.** *Assume that* $u_0 \in \dot{H}^1(\mathbb{R}^2) \cap \dot{H}^2(\mathbb{R}^2)$ *and* $u_1 \in H^1(\mathbb{R}^2)$. *Then there exists* $T > 0$, $T = T(\|u_0\|_{\dot{H}^1(\mathbb{R}^2)}, \|u_0\|_{\dot{H}^2(\mathbb{R}^2)}, \|u_1\|_{H^1(\mathbb{R}^2)})$, *such that (2.1) has a unique solution* u *satisfying*

$$u \in C(0,T;\dot{H}^2(\mathbb{R}^2) \cap \dot{H}^1(\mathbb{R}^2)) \cap L^r(0,T;\dot{H}_q^{2\sigma-1}(\mathbb{R}^2)),$$

$$u_t \in C(0,T;H^1(\mathbb{R}^2)) \cap L^r(0,T;\dot{H}_q^{2\sigma-2}(\mathbb{R}^2)),$$

$$\nabla u, u_t \in C(0,T;H^1(\mathbb{R}^2)) \cap L^4(0,T;L^\infty(\mathbb{R}^2)), \tag{2.2}$$

with $r = \dfrac{4q}{q-2}$, $\sigma = \dfrac{9}{8} + \dfrac{3}{4q}$ *and* $2 < q < \infty$.

Moreover, for all $0 < T' < T$ *there exists a neighborhood* V *of* $(u_0, u_1) \in \dot{H}^1(\mathbb{R}^2) \cap \dot{H}^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ *such that the map data solution*

$$\begin{array}{ccc} V & \rightarrow & C(0,T';\dot{H}^2(\mathbb{R}^2) \cap \dot{H}^1(\mathbb{R}^2)) \cap L^r(0,T';\dot{H}_q^{2\sigma-1}(\mathbb{R}^2)) \\ (\tilde{u}_0, \tilde{u}_1) & \rightarrow & \tilde{u}(t) \end{array}$$

is Lipschitz.

Remark 2.1.3. *It is important to observe that the flow of (2.1) preserves the Hamiltonian*

$$\begin{aligned} H(u)(t) = & \|\partial_t u(t)\|_2^2 + \mu\ b\|\partial_t u(t)\|_{\dot{H}^1}^2 \\ & + \|u(t)\|_{\dot{H}^1}^2 + \mu\ a\|u(t)\|_{\dot{H}^2}^2 = H(u)(0). \end{aligned} \tag{2.3}$$

See [24] for the proof of (2.3).

Using the previous remark, it is possible to establish an *a priori* estimate to prove the following global result for (BL) in the energy space.

Corollary 2.1.4 (Global well-posedness)**.** *For any* $T > 0$, $u_0 \in \dot{H}^1(\mathbb{R}^2) \cap \dot{H}^2(\mathbb{R}^2)$ *and* $u_1 \in H^1(\mathbb{R}^2)$ *there exists a unique solution* u *of (2.1) such that*

$$\nabla u \in C(0,T;H^1(\mathbb{R}^2)),\ \partial_t u \in C(0,T;H^1(\mathbb{R}^2)).$$

And the solution u *has local regularity*

$$u \in L^r(0,T;\dot{H}_q^{2\sigma-1}(\mathbb{R}^2)),$$

$$u_t \in L^r(0,T;\dot{H}_q^{2\sigma-2}(\mathbb{R}^2)),$$

with $r = \dfrac{4q}{q-2}$, $\sigma = \dfrac{9}{8} + \dfrac{3}{4q}$ *and* $2 < q < \infty$.

We also consider the well-posedness for the Cauchy problem of the isotropic Benney-Luke equation (2.1) in three spatial dimensions. In this case we have the following.

Theorem 2.1.5 (Local well-posedness and local regularity)**.** *Assume that* $(u_0, u_1) \in H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$ *and* $2 < s \leqslant 5/2$. *Then there exists* $T > 0$ *such that the Cauchy problem:*

$$\begin{cases} (1 - bc^{-2}\Delta)(u_{tt} - \Delta u) = c^{-2}(1 - c^2)\Delta u - c^{-2}\left(u_t\Delta u + 2\nabla u \cdot \nabla u_t\right), \\ u(0, \mathbf{x}) = u_0(\mathbf{x}), \quad u_t(0, \mathbf{x}) = u_1(\mathbf{x}), \end{cases} \tag{2.4}$$

has a unique solution u *satisfying*

$$u \in C(0, T; H^s(\mathbb{R}^3)), \quad \partial_t u \in C(0, T; H^{s-1}(\mathbb{R}^3)),$$

$$\left(\int_0^T \|(-\Delta)^{\sigma-1/2}u(t,\cdot)\|_q^r dt\right)^{1/r} < \infty, \; \left(\int_0^T \|(-\Delta)^{\sigma-1}u_t(t,\cdot)\|_q^r dt\right)^{1/r} < \infty \tag{2.5}$$

and

$$\int_0^T \|(\nabla u, u_t)(t)\|_\infty^2 dt < \infty \tag{2.6}$$

with $r = \dfrac{2q}{q-2}$, $\sigma = \dfrac{s}{2} + \dfrac{1}{q}$ *and* $(s-2)^{-1} < q < \infty$.

Where u *is such that*

$$\Phi(t, \mathbf{x}) := \frac{\sqrt{\mu}}{\epsilon} u\left(\frac{t}{\sqrt{\mu}}, \frac{1}{c\sqrt{\mu}}\,\mathbf{x}\right),$$

with Φ *satisfying the isotropic Benney-Luke equation (1),* $c^2 = a/b$, $(t, \mathbf{x}) \in \mathbb{R}_+ \times \mathbb{R}^3$, $\mathbb{R}_+ = [0, \infty)$, *$a$ and b are positive real constants and* ∇ *and* Δ *are the three-dimensional gradient and Laplacian, respectively.*

2.2 Proof of Theorem 2.1.1

We will use the fixed point theorem and the Strichartz estimates. We begin by rewritten the equation of the IVP (2.1) in the equivalent form

$$(1 - bc^{-2}\Delta)(u_{tt} - \Delta u) = c^{-2}(1 - c^2)\Delta u - c^{-2}[\Delta, u]u_t, \tag{2.7}$$

where $[\Delta, u]u_t := \Delta(uu_t) - u\Delta u_t$.

We will use the next notation $G(u) = G_1(u) + G_2(u)$ where

$$G_1(u) = c^{-2}(1 - c^2)\Delta(1 - bc^{-2}\Delta)^{-1}u \tag{2.8}$$

and

$$G_2(u) = -c^{-2}(1 - bc^{-2}\Delta)^{-1}[\Delta, u]u_t. \tag{2.9}$$

Then we can write the solution of the IVP associated to (2.7) as

$$u(t) = \dot{\mathbf{K}}(t)u_0 + K(t)u_1 + \int_0^t \mathbf{K}(t - t')G(u)(t')dt'. \tag{2.10}$$

We prove the local well-posedness for the IVP associated to (2.7) in $H^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$. To do so, we will use a fixed point argument as above mentioned.

For $M, T > 0$ and $2 < q < \infty$, define the complete metric space

$$X_T^M = \{u \in C([0, T]; H^2(\mathbb{R}^2)) : |||u||| \leq M\}$$

where

$$|||u||| = \|u\|_{L_T^\infty H^2} + \|u_t\|_{L_T^\infty H^1} + \|u\|_{L_T^r H_q^{2\sigma-1}} + \|u_t\|_{L_T^r H_q^{2\sigma-2}}, \tag{2.11}$$

with $r = \dfrac{4q}{q-2}$ and $\sigma = \dfrac{9}{8} + \dfrac{3}{4q}$.

We shall prove that for an appropriate choice of T and M the operator

$$\mathbb{F}(u)(t) = \dot{\mathbf{K}}(t)u_0 + K(t)u_1 + \int_0^t \mathbf{K}(t-t')G(u)(t')dt' \tag{2.12}$$

is a contraction on X_T^M.

We estimate $\|\mathbb{F}(u)\|_{H^2}$ and $\|\partial_t\mathbb{F}(u)\|_{H^1}$ using the linear estimate (1.6), as follows,

$$\begin{aligned}\|\mathbb{F}(u)(t)\|_{H^2}+&\|\partial_t\mathbb{F}(u)(t)\|_{H^1} \lesssim \|u_0\|_{H^2} + (1+t)\|u_1\|_{H^1} \\ &+ \int_0^t (t-t')\|G(u)(t')\|_2 dt' + \int_0^t \|(-\Delta)^{\frac{1}{2}}G(u)(t')\|_2 dt'.\end{aligned} \tag{2.13}$$

Note that

$$\|(-\Delta)^{(s-1)/2}G_2(u)(t)\|_2 \lesssim \|[\Delta, u]u_t(t)\|_2 \tag{2.14}$$

if $1 \le s \le 3$, $n \ge 1$.

Using (2.14), Lemma 1.3.4 and Hölder's inequality we have

$$\begin{aligned} I_1 :=& \int_0^t (t-t')\|G(u)(t')\|_2 dt' \\ \le& C\int_0^t (t-t')\|u_t\|_{H^1}\left(\|u\|_{\dot{H}^{s_0+1}} + \|(-\Delta)^{\sigma-\frac{1}{2}}u\|_q\right)dt' \\ &+ C\int_0^t (t-t')\|u\|_{\dot{H}^2}\left(|1-c^2| + \|u_t\|_{H^1} + \|(-\Delta)^{\sigma-1}u_t\|_q\right)dt' \\ \lesssim& 2\,T^2\ \|u\|_{L_T^\infty H^2}\|u_t\|_{L_T^\infty H^1} + |1-c^2|\ T^2\ \|u\|_{L_T^\infty H^2} \\ &+ \|u_t\|_{L_T^\infty H^1}\Big(\int_0^t (t-t')^{r'}dt'\Big)^{\frac{1}{r'}}\Big(\int_0^t \|(-\Delta)^{\sigma-\frac{1}{2}}u\|_q^r\, dt'\Big)^{1/r} \\ &+ \|u\|_{L_T^\infty H^2}\Big(\int_0^t (t-t')^{r'}dt'\Big)^{\frac{1}{r'}}\Big(\int_0^t \|(-\Delta)^{\sigma-1}u_t\|_q^r\, dt'\Big)^{1/r}. \end{aligned} \tag{2.15}$$

Then

$$\begin{aligned} I_1 \lesssim& |1-c^2|\ T^2\ \|u\|_{L_T^\infty H^2} +\ T^2\ \|u\|_{L_T^\infty H^2}\|u_t\|_{L_T^\infty H^1} + \\ & T^\beta\{\|u_t\|_{L_T^\infty H^1}\ \|u\|_{L_T^r \dot{H}_q^{2\sigma-1}} + \|u\|_{L_T^\infty H^2}\ \|u_t\|_{L_T^r \dot{H}_q^{2\sigma-2}}\}, \end{aligned} \tag{2.16}$$

where $\beta = 1 + \frac{1}{r'} > \frac{7}{4}$.

Again using (2.14), Lemma 1.3.4, (1.33) and Hölder's inequality we have

$$\begin{aligned} I_2 := &\int_0^t \|(-\Delta)^{\frac{1}{2}} G(u)(t')\|_2 dt' \\ &\lesssim T \, \|u\|_{L_T^\infty \dot{H}^2} \|u_t\|_{L_T^\infty H^1} + T|1-c^2| \, \|u\|_{L_T^\infty \dot{H}^1} \\ &\quad + T\|u_t\|_{L_T^\infty \dot{H}^1} \|u\|_{L_T^\infty \dot{H}^1}^{1-s_0} \|u\|_{L_T^\infty \dot{H}^2}^{s_0} \\ &\quad + T^{\beta-1}\{\|u_t\|_{L_T^\infty H^1} \, \|u\|_{L_T^r \dot{H}_q^{2\sigma-1}} + \|u\|_{L_T^\infty \dot{H}^2} \, \|u_t\|_{L_T^r \dot{H}_q^{2\sigma-2}}\}. \end{aligned} \tag{2.17}$$

Then using (2.13), (2.16) and (2.17) it follows that

$$\begin{aligned} \|\mathbb{F}(u)(t)\|_{L_T^\infty H^2} + \|\partial_t \mathbb{F}(u)\|_{L_T^\infty H^1} \leq &C\|u_0\|_{H^2} + C(1+T)\|u_1\|_{H^1} \\ &+ C\left(|1-c^2|T^2 + P(T)|||u|||\right)|||u||| \end{aligned} \tag{2.18}$$

where $P(T) = T^\beta + T^2 + T^{\beta-1} + T$.

Now we want to estimate the mixed norms. First, we recall that

$$\sigma - 1 = \frac{3}{4q} + \frac{1}{8}, \qquad r = \frac{4q}{q-2}, \qquad 2 < q < \infty$$

and $n = 2$. If $r_1 = r,\ q_1 = q,\ r_2 = \infty,\ q_2 = 2,\ \rho_1 = 2\sigma - 1,\ \mu = 2$ and $-\rho_2 = 1$ then $r_i,\ q_i,\ \rho_i$ and $\mu,\ i = 1, 2$, satisfy (1.17), (1.18), (1.19) and (1.20) and using the Strichartz estimates (1.21) and (1.22) we have

$$\begin{aligned} I_3 = &\|\mathbb{F}(u)\|_{L_T^r \dot{H}_q^{2\sigma-1}} + \|\partial_t \mathbb{F}(u)\|_{L_T^r \dot{H}_q^{2\sigma-2}} \\ \leq &C\left(\|u_0\|_{\dot{H}^2} + \|u_1\|_{\dot{H}^1} + \int_0^T \|(-\Delta)^{\frac{1}{2}} G(u)(t')\|_2 dt'\right). \end{aligned} \tag{2.19}$$

Using the same estimates obtained for I_2 we have

$$I_3 \qquad \lesssim \|u_0\|_{H^2} + (1+T)\|u_1\|_{H^1} \tag{2.20}$$

$$+T|1-c^2|\|u\|_{L_T^\infty \dot{H}^1} + P(T)|||u|||^2. \tag{2.21}$$

Putting together the estimates (2.18) and (2.20) it follows that

$$\begin{aligned} |||\mathbb{F}(u)||| \leq &C\left(\, \|u_0\|_{H^2} + (1+T)\|u_1\|_{H^1}\right) \\ &+ C\left(|1-c^2|(T+T^2) \ + P(T)|||u|||\right)|||u|||. \end{aligned} \tag{2.22}$$

Let $\delta = \|u_0\|_{H^2} + \|u_1\|_{H^1}$, $M = 2C(1+T)\delta$ and T such that

$$C|1-c^2|(T+T^2) + C(T^2 + T^\beta + T + T^{\beta-1})M \leq 1/2 \tag{2.23}$$

then we have that $\mathbb{F}(X_T^M) \subset X_T^M$.

Noticing that

$$\begin{aligned} |||\mathbb{F}(u) - \mathbb{F}(\tilde{u})||| \lesssim & \int_0^T (T-t')\|G(u)(t') - G(\tilde{u})(t')\|_2 dt' \\ & + \int_0^T \|(-\Delta)^{1/2}\left(G(u)(t') - G(\tilde{u})(t')\right)\|_2 dt', \end{aligned} \tag{2.24}$$

and using the fact that

$$G_2(u) - G_2(\tilde{u}) = -c^{-2}(1-bc^{-2}\Delta)^{-1}([\Delta,(u-\tilde{u})]\partial_t u + [\Delta,\tilde{u}](u-\tilde{u})_t) \tag{2.25}$$

it follows from Lemma 1.3.4 that if $u, \tilde{u} \in X_M^T$ then

$$|||\mathbb{F}(u) - \mathbb{F}(\tilde{u})||| \lesssim \left((T+T^2)|1-c^2| + (T+T^{1/r'})M\right) |||u-\tilde{u}|||. \tag{2.26}$$

Thus, there exists a unique fixed point of $\mathbb{F}$ which is a solution of the integral equation (2.12) if $(T+T^2)|1-c^2| + (T+T^{1/r'})M < 1$. $\square$

2.3 Proof of Theorem 2.1.2

Fix q, $2 < q < \infty$. Let $u_0 \in \dot{H}^1(\mathbb{R}^2) \cap \dot{H}^2(\mathbb{R}^2)$, $u_1 \in H^1(\mathbb{R}^2)$ and

$$\mathbb{F}_{(u_0,u_1)}(u)(t) = \mathbb{F}(u)(t) = \dot{\mathbf{K}}(t)u_0 + \mathbf{K}(t)u_1 + \int_0^t \mathbf{K}(t-t')G(u)(t')dt'. \tag{2.27}$$

We define the complete metric space

$$Y_T^M = \{u \in C([0,T]; \dot{H}^2(\mathbb{R}^2) \cap \dot{H}^1(\mathbb{R}^2)) : |||u|||_Y \leq M\}$$

with

$$\begin{aligned} |||u|||_Y = & \|u\|_{L_T^\infty \dot{H}^2} + \|u\|_{L_T^\infty \dot{H}^1} + \|u_t\|_{L_T^\infty H^1} \\ & + \left(\int_0^T \|(-\Delta)^{\sigma-1/2}u(t)\|_q^r dt\right)^{1/r} + \left(\int_0^T \|(-\Delta)^{\sigma-1}u_t(t)\|_q^r dt\right)^{1/r}, \end{aligned} \tag{2.28}$$

$r = \dfrac{4q}{q-2}$, $\sigma = \dfrac{9}{8} + \dfrac{3}{4q}$.

We shall prove that for an appropriate choice of T and M the operator given by (2.27) is a contraction on Y_T^M.

Let $r_1 = r,\ q_1 = q,\ r_2 = \infty,\ q_2 = 2,\ \rho_1 = 2\sigma - 1,\ \mu = 2$ and $-\rho_2 = 1$, then $r_i,\ q_i,\ \rho_i$ and $\mu,\ i = 1,2$, satisfy (1.17), (1.18), (1.19), (1.20). Now, using the linear estimate (1.6), the Strichartz estimates (1.21) and (1.22) with $r_i,\ q_i,\ \rho_i$ and $\mu,\ i = 1,2$ we get

$$\begin{aligned}|||\mathbb{F}(u)|||_Y \lesssim& \|u_0\|_{\dot{H}^2} + \|u_0\|_{\dot{H}^1} \\ &+ (1+T)\|u_1\|_{H^1} + \int_0^T \|(-\Delta)^{\frac{1}{2}} G(u)(t')\|_{L^2} dt'.\end{aligned} \tag{2.29}$$

From (2.17) we have

$$\begin{aligned}|||\mathbb{F}(u)|||_Y \leq& C\left(\|u_0\|_{\dot{H}^2} + \|u_0\|_{\dot{H}^1} + (1+T)\|u_1\|_{H^1}\right) \\ &+ C\left(|1-c^2|T \ + (T^{\beta-1} + T)|||u|||_Y\right)|||u|||_Y,\end{aligned} \tag{2.30}$$

where $\beta = 1 + \dfrac{1}{r'} > \dfrac{7}{4}$.

Let $\delta = \|u_0\|_{\dot{H}^2} + \|u_0\|_{\dot{H}^1} + \|u_1\|_{H^1}$, $M = 2C(1+T)\delta$ and T such that

$$C|1-c^2|T + C(T + T^{\beta-1})M \leq 1/2 \tag{2.31}$$

then we have that $\mathbb{F}(Y_T^M) \subset Y_T^M$.

Since

$$|||\mathbb{F}(u) - \mathbb{F}(\tilde{u})|||_Y \lesssim \int_0^T \|(-\Delta)^{1/2}\left(G(u)(t') - G(\tilde{u})(t')\right)\|_2 dt'. \tag{2.32}$$

Using (2.25) and Lemma 1.3.4 we get

$$|||\mathbb{F}(u) - \mathbb{F}(\tilde{u})|||_Y \leq C\left(|1-c^2|T \ + (T^{\beta-1} + T)M\right)|||u-\tilde{u}|||_Y \tag{2.33}$$

whenever $u, \tilde{u} \in Y_T^M$.

Then, there exists a unique fixed point of $\mathbb{F}$ if

$$C(|1-c^2|T \ + (T^{\beta-1} + T)M) < 1.$$

Therefore, the existence and uniqueness of the solution of the problem (2.1) have been proved in the metric space Y_T^M. The uniqueness of the solution in the space $\dot{H}^1(\mathbb{R}^2) \cap \dot{H}^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ is obtained by standard arguments.

Using similar arguments to the those applied in the continuous dependence proof in Theorem 2.1.1 one can show that the map data solution is locally Lipschitz. □

2.4 Proof of Corollary 2.1.4

Now we will show that the local solution obtained in Theorem 2.1.2 can be extended to $[0, T]$, for any $T > 0$, time interval. It suffices to prove the existence of a uniform bound for $\|u(t)\|^2_{\dot{H}^1}$, $\|u(t)\|^2_{\dot{H}^2}$, $\|\partial_t u(t)\|^2_2$ and $\|\partial_t u(t)\|^2_{\dot{H}^1}$. This allows us to establish an *a priori* estimate and then make use of the local theory to extend the solution. To do so we use the following conserved quantity

$$\begin{aligned} H(u)(t) &= \|\partial_t u(t)\|_2^2 + \mu\ b\|\partial_t u(t)\|^2_{\dot{H}^1} + \|u(t)\|^2_{\dot{H}^1} + \mu\ a\|u(t)\|^2_{\dot{H}^2} \\ &= H(u)(0), \end{aligned} \tag{2.34}$$

satisfied by the flow of (11) for $p \geq 1$ integer (see [29]).

2.5 Proof of Theorem 2.1.5 (3-dimensional case)

Fix s, $2 < s \leq 5/2$ and take $q \in (1/(s-2), \infty)$. For $T, M > 0$ define the complete metric space

$$X_T^M = \{u \in C(0,T; H^s(\mathbb{R}^3)) : |||u||| \leq M\}$$

where

$$\begin{aligned} |||u||| =& \|u\|_{L_T^\infty H^s(\mathbb{R}^3)} + \|u_t\|_{L_T^\infty H^{s-1}(\mathbb{R}^3)} \\ &+ \|u\|_{L_T^r \dot{H}_q^{2\sigma-1}(\mathbb{R}^3)} + \|u_t\|_{L_T^r \dot{H}_q^{2\sigma-2}(\mathbb{R}^3)}, \end{aligned} \tag{2.35}$$

with $r = \dfrac{2q}{q-2}$, $\sigma = \dfrac{s}{2} + \dfrac{1}{q}$.

Let the operator

$$\mathbb{F}(u)(t) = \dot{\mathbf{K}}(t)u_0 + K(t)u_1 + \int_0^t \mathbf{K}(t-t')G(u)(t')dt'. \tag{2.36}$$

It is possible to prove that $\mathbb{F}$ is a contraction in X_T^M using the similar arguments as in the proof Theorem 2.1.1.

Using the Strichartz estimates (1.21) and (1.22) with $r_1 = r,\ q_1 = q,\ r_2 = \infty,\ q_2 = 2,\ \rho_1 = 2\sigma - 1,\ \mu = s$ and $-\rho_2 = s - 1$ and the linear estimate we get

$$\begin{aligned} |||\mathbb{F}(u)||| \lesssim& \|u_0\|_{H^s(\mathbb{R}^3)} + (1+T)\|u_0\|_{H^{s-1}(\mathbb{R}^3)} \\ &+ \int_0^T (T-t')\|G(u)(t')\|_{L^2(\mathbb{R}^3)}dt' \\ &+ \int_0^T \|(-\Delta)^{\frac{s-1}{2}} G(u)(t')\|_{L^2(\mathbb{R}^3)}dt'. \end{aligned} \tag{2.37}$$

For (2.14), Lemma 1.3.7 and Hölder's inequality we have

$$\int_0^T (T-t')\|G(u)(t')\|_{L^2(\mathbb{R}^3)}dt' \lesssim |1-c^2|\ T^2\ \|u\|_{L^\infty_T H^s(\mathbb{R}^3)}$$
$$+\ T^2\,\|u\|_{L^\infty_T H^s(\mathbb{R}^3)}\|u_t\|_{L^\infty_T H^{s-1}(\mathbb{R}^3)}$$
$$+\ T^\beta\|u_t\|_{L^\infty_T H^{s-1}(\mathbb{R}^3)}\ \|u\|_{L^r_T \dot{H}^{2\sigma-1}_q(\mathbb{R}^3)}$$
$$+\ T^\beta\|u\|_{L^\infty_T H^s(\mathbb{R}^3)}\ \|u_t\|_{L^r_T \dot{H}^{2\sigma-2}_q(\mathbb{R}^3)}, \tag{2.38}$$

and

$$\int_0^t \|(-\Delta)^{\frac{s-1}{2}} G(u)(t')\|_2 dt' \lesssim T\ \|u\|_{L^\infty_T H^s(\mathbb{R}^3)}\|u_t\|_{L^\infty_T H^{s-1}}$$
$$+\ T|1-c^2|\ \|u\|_{L^\infty_T H^{s-1}}$$
$$+\ T\|u_t\|_{L^\infty_T H^{s-1}}\|u\|_{L^\infty_T H^s}$$
$$+\ T^{\beta-1}\|u_t\|_{L^\infty_T H^{s-1}}\ \|u\|_{L^r_T \dot{H}^{2\sigma-1}_q}$$
$$+\ T^{\beta-1}\|u\|_{L^\infty_T \dot{H}^2}\ \|u_t\|_{L^r_T \dot{H}^{2\sigma-2}_q}. \tag{2.39}$$

with $\beta = 1 + \dfrac{1}{r'} > 1$.

Putting together the estimates (2.38) and (2.39) it follows that

$$|||\mathbb{F}(u)||| \le C\,(\ \|u_0\|_{H^s} + (1+T)\|u_1\|_{H^{s-1}})$$
$$+\, C|1-c^2|(T+T^2)|||u|||$$
$$+\, C(T^\beta + T^2 + T^{\beta-1} + T)|||u|||^2. \tag{2.40}$$

Let $\delta = \|u_0\|_{H^s} + \|u_1\|_{H^{s-1}}$, $M = 2C(1+T)\delta$ and T such that

$$C|1-c^2|(T+T^2) + C(T^2 + T^\beta + T + T^{\beta-1})M \le 1/2. \tag{2.41}$$

Then we have that $\mathbb{F}(X^M_T) \subset X^M_T$.

Using that

$$|||\mathbb{F}(u) - \mathbb{F}(\tilde{u})||| \lesssim \int_0^T (T-t')\|G(u)(t') - G(\tilde{u})(t')\|_{L^2(\mathbb{R}^3)}dt'$$
$$+ \int_0^T \|(-\Delta)^{(s-1)/2}\,(G(u)(t') - G(\tilde{u})(t'))\,\|_{L^2(\mathbb{R}^3)}dt'. \tag{2.42}$$

It follows from (2.25), Lemma 1.3.7 that if $u, \tilde{u} \in X_T^M$ then

$$|||\mathbb{F}(u) - \mathbb{F}(\tilde{u})||| \lesssim \left((T + T^2)|1 - c^2| + (T + T^{1/r'})M\right) |||u - \tilde{u}|||. \quad (2.43)$$

Thus, $\mathbb{F}$ is a contraction in X_T^M under the restriction in (2.41) for M and T. Therefore, exists a unique solution of (2.4) in X_T^M, but for standards arguments we have the uniqueness of solution in the space $H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$.

□

Chapter 3

Isotropic p-generalized Benney-Luke equation: well-posedness results and local regularity

3.1 Introduction and statements of the results

The isotropic p-generalized Benney-Luke equations is giving by

$$\Phi_{tt} - \Delta\Phi + \mu(a\Delta^2\Phi - b\Delta\Phi_{tt}) + \epsilon(\Phi_t\Delta_p\Phi + 2\nabla^p\Phi \cdot \nabla\Phi_t) = 0, \qquad (3.1)$$

where ∇^p and Δ_p are

$$\nabla^p\Phi = ((\partial_x\Phi)^p, (\partial_y\Phi)^p) \qquad (3.2)$$

$$\Delta_p\Phi = \nabla \cdot (\nabla^p\Phi) = \partial_x(\partial_x\Phi)^p + \partial_y(\partial_y\Phi)^p. \qquad (3.3)$$

In this chapter we study the local regularity of the isotropic p-generalized Benney-Luke and the local well-posedness in the energy space, $\dot{H}^2(\mathbb{R}^2) \cap \dot{H}^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$. The main result is the global well-posedness in this space. We remind that, the natural space for the solitary wave of IVP associated to (p-gBL) equations is the energy space.

We prove the local well-posedness for isotropic p-generalized Benney-Luke equations using a fixed point argument and the generalized Strichartz inequalities for the wave equation.

We define u such that

$$\Phi(t, \mathbf{x}) := \frac{\sqrt{\mu}}{\sqrt[p]{\epsilon}} u\left(\frac{t}{\sqrt{\mu}}, \frac{1}{\sqrt{\mu}}\mathbf{x}\right), \tag{3.4}$$

with Φ satisfying the isotropic p-generalized Benney-Luke equation, $p \in \mathbb{Z}^+$ and $p > 1$, then the associated initial value problem (3.1) is equivalent to

$$\begin{cases} (1 - b\Delta)(u_{tt} - c^2\Delta u) = (1 - c^2)\Delta u - F_p(u_t, \nabla^p u, \nabla u_t) \\ u(0, \mathbf{x}) = u_0(\mathbf{x}) \quad u_t(0, \mathbf{x}) = u_1(\mathbf{x}), \end{cases} \tag{3.5}$$

where $c^2 = \dfrac{a}{b}$, $u_i(\mathbf{x}) = \dfrac{\sqrt[p]{\epsilon}}{\sqrt{\mu}}\Phi_i(\sqrt{\mu}\,\mathbf{x})$, $i = 0, 1$, $\mathbf{x} \in \mathbb{R}^2$ and

$$\begin{aligned} F_p(u_t, \nabla^p u, \nabla u_t) &= p\, u_t(u_x)^{p-1}u_{xx} + p\, u_t u(u_y)^{p-1}u_{yy} \\ &\quad + 2\partial_t u_x(u_x)^p + 2\partial_t u_y(u_y)^p, \end{aligned} \tag{3.6}$$

(notice that F_p is the nonlinear term of isotropic Benney-Luke equation when $p = 1$). The nonlinear term

$$(1 - b\Delta)^{-1}F_p(u_t, \nabla^p u, \nabla u_t)$$

does not satisfy a "null condition" but it is possible to prove that the Sobolev exponent $s = 2$ can be achieved in two dimensions.

Using the Strichartz estimates we prove that the (p-gBL) equation is locally and globally well-posed in the energy space and we establish local regularity results.

Theorem 3.1.1. *Assume that $p \geq 2$ integer and $u_0 \in \dot{H}^1(\mathbb{R}^2) \cap \dot{H}^2(\mathbb{R}^2)$, $u_1 \in H^1(\mathbb{R}^2)$. Then there exist $T = T(\|u_0\|_{\dot{H}^1(\mathbb{R}^2)}, \|u_0\|_{\dot{H}^2(\mathbb{R}^2)}, \|u_1\|_{H^1(\mathbb{R}^2)}) > 0$*

and a unique solution u of (3.5) such that

$$u \in C(0,T;\dot{H}^2(\mathbb{R}^2)\cap \dot{H}^1(\mathbb{R}^2))\cap L^r(0,T;\dot{H}^{2\sigma-1}_q(\mathbb{R}^2)),$$

$$u_t \in C(0,T;H^1(\mathbb{R}^2))\cap L^r(0,T;\dot{H}^{2\sigma-2}_q(\mathbb{R}^2)).$$

In addition u has the following local regularity

$$\left(\int_0^T \|(-\Delta)^{\sigma-1/2}u(t,\cdot)\|^r_{L^q}\,dt\right)^{1/r} < \infty,$$

$$\left(\int_0^T \|(-\Delta)^{\sigma-1}u_t(t,\cdot)\|^r_{L^q}\,dt\right)^{1/r} < \infty,$$

with $r = \dfrac{4q}{q-2}$, $\sigma = \dfrac{9}{8}+\dfrac{3}{4q}$ *and* $2 < q < q(p)$, *where*

$$q(p) = \begin{cases} \infty, & p = 2,3,4,\\ \dfrac{2p}{p-4}, & p > 4. \end{cases} \tag{3.7}$$

Moreover, for all $0 < T' < T$ *there exists a neighborhood* V *of* $(u_0,u_1) \in \dot{H}^1(\mathbb{R}^2)\cap \dot{H}^2(\mathbb{R}^2)\times H^1(\mathbb{R}^2)$ *such that the map data solution*

$$\begin{array}{ccc} V & \to & C(0,T';\dot{H}^2(\mathbb{R}^2)\cap \dot{H}^1(\mathbb{R}^2))\cap L^r(0,T';\dot{H}^{2\sigma-1}_q(\mathbb{R}^2)) \\ (\tilde{u}_0,\tilde{u}_1) & \to & \tilde{u}(t) \end{array}$$

is Lipschitz.

Remark 3.1.2. *The p-generalized Benney-Luke equations (3.1) have a conserved quantity as the Benney-Luke equation, i.e.,*

$$\begin{aligned} H(\Phi)(t) &= \|\Phi_t(t)\|_2^2 + \mu\, b\|\Phi_t(t)\|^2_{\dot{H}^1} + \|\Phi(t)\|^2_{\dot{H}^1} + \mu\, a\|\Phi(t)\|^2_{\dot{H}^2} \\ &= H(\Phi)(0). \end{aligned} \tag{3.8}$$

Proof. See [29].

Corollary 3.1.3. *Let $p \geq 2$ integer and $T > 0$. Then for all the functions u_0, u_1 such that $u_0 \in \dot{H}^1(\mathbb{R}^2) \cap \dot{H}^2(\mathbb{R}^2)$, $u_1 \in H^1(\mathbb{R}^2)$, there exists a unique solution u of (3.5) such that*

$$\nabla u \in C(0, T; H^1(\mathbb{R}^2)), \;\; u_t \in C(0, T, H^1(\mathbb{R}^2)),$$

and

$$\int_0^T \|(\nabla u, u_t)(\cdot, t)\|_{L^\infty}^4 \, dt < \infty.$$

3.2 Proof of Theorem 3.1.1

The tools we will use to show this are: Fixed point theorem, generalized Stricharzt estimates for the wave equation and Lemma 1.3.3.

Using the scale change $\tilde{\mathbf{x}} = c\,\mathbf{x}$ and denoting the new function with the same variable we have the following equivalent equation for (p-gBL)

$$u_{tt} - \Delta u = B^{-1} G(u) \tag{3.9}$$

with

$$\begin{aligned} &G(u) = G_0(u) + G_p(u) \\ &G_0(u) = (1 - c^2) c^{-2} \Delta u = m_2 \Delta u \\ &G_p(u) = -c^{-(p+1)} F_p(u_t, \nabla^p u, \nabla u_t) = k_p F_p(u_t, \nabla^p u, \nabla u_t) \\ &B\phi = (1 - m_1^2 \Delta)\phi, \quad m_1^2 = \frac{b^2}{a}. \end{aligned} \tag{3.10}$$

Fix $p \geq 2$ integer, $2 < q < q(p)$, where $q(p)$ is giving by (3.7) and let $r = \dfrac{4q}{q-2}$, $\sigma = \dfrac{9}{8} + \dfrac{3}{4q}$ and for $T, M > 0$ define the complete metric space

$$X_T^M = \{u \in C([0,T]; \dot{H}^1(\mathbb{R}^2) \cap \dot{H}^2(\mathbb{R}^2)) : |||u|||_X \leq M\},$$

where

$$|||u|||_X = \|u\|_{L^\infty_T \dot{H}^1} + \|u\|_{L^\infty_T \dot{H}^2} + \|u_t\|_{L^\infty_T H^1} + \|u\|_{L^r_T \dot{H}^{2\sigma-1}_q} + \|u_t\|_{L^r_T \dot{H}^{2\sigma-2}_q}, \tag{3.11}$$

and let

$$\mathbb{F}(u) = \dot{K}(t)u_0 + K(t)u_1 + \int_0^t K(t-t')B^{-1}G(u)(t')\,dt', \tag{3.12}$$

where G and B are giving by (3.10).

Using the linear estimates for $K(t)$ and $\dot{K}(t)$ we have

$$\|\mathbb{F}(u)(t)\|_{\dot{H}^1} + \|\mathbb{F}(u)(t)\|_{\dot{H}^2} + \|\partial_t \mathbb{F}(u)(t)\|_{H^1} \lesssim \|u_0\|_{\dot{H}^1} + \|u_0\|_{\dot{H}^2} + (1+t)\|u_1\|_{H^1} + \int_0^t \|(-\Delta)^{\frac{1}{2}} B^{-1}G(u)(t')\|_2\,dt' \tag{3.13}$$

and the Strichartz estimates imply

$$\|\mathbb{F}(u)\|_{L^r_T \dot{H}^{2\sigma-1}_q} + \|\partial_t \mathbb{F}(u)\|_{L^r_T \dot{H}^{2\sigma-2}_q} \lesssim \|u_0\|_{\dot{H}^2} + \|u_1\|_{H^1} + \int_0^T \|(-\Delta)^{\frac{1}{2}} B^{-1}G(u)(t')\|_2\,dt' \tag{3.14}$$

then

$$|||\mathbb{F}(u)|||_X \lesssim (1+T)(\|u_0\|_{\dot{H}^1} + \|u_0\|_{\dot{H}^2} + \|u_1\|_{H^1}) + \int_0^T \|(-\Delta)^{\frac{1}{2}} B^{-1}G(u)(t')\|_2\,dt'. \tag{3.15}$$

Remark 3.2.1.

$$\begin{aligned} \|(-\Delta)^{\frac{1}{2}} B^{-1}G(u)(t')\|_2 = & \ \|(-\Delta)^{\frac{1}{2}} B^{-1}(m_2\Delta u + k_p F_p(u_t, \nabla^p u, \nabla u_t)(t'))\|_2 \\ \lesssim & \ |m_2|\|u(t')\|_{\dot{H}^1} + |k_p|\|F_p(u_t, \nabla^p u, \nabla u_t)(t'))\|_2. \end{aligned}$$

This remark and (3.15) imply that

$$|||\mathbb{F}(u)|||_X \lesssim (1+T)(\|u_0\|_{\dot{H}^1} + \|u_0\|_{\dot{H}^2} + \|u_1\|_{H^1}) + T|m_2|\|u\|_{L^\infty_T \dot{H}^1} + |k_p| \int_0^T \|F_p(u)\|_2\,dt'. \tag{3.16}$$

We have from (3.6)

$$I_4 := \int_0^T \|F_p(u)\|_2 \, dt' \lesssim p \int_0^T \|u_t(t')\|_\infty \|\nabla u(t')\|_\infty^{p-1} \|u(t')\|_{\dot H^2} \, dt' + 2\int_0^T \|\nabla u(t')\|_\infty^p \|u_t(t')\|_{\dot H^1} \, dt',$$

from Lemma 1.3.3

$$\begin{aligned} I_4 \lesssim p \int_0^T & \left\{ \|u_t(t')\|_{\dot H^{s_0}} + \|u_t(t')\|_{\dot H_q^{2\sigma-2}} \right\} \times \\ & \left\{ \|u(t')\|_{\dot H^{s_0+1}} + \|u(t')\|_{\dot H_q^{2\sigma-1}} \right\}^{p-1} \|u(t')\|_{\dot H^2} \, dt' \\ & + 2\|u_t\|_{L_T^\infty \dot H^1} \int_0^T \left(\|u(t')\|_{\dot H^{s_0+1}} + \|u(t')\|_{\dot H_q^{2\sigma-1}} \right)^p dt'. \end{aligned}$$

Therefore

$$\begin{aligned} I_4 \lesssim{} & p \, \|u_t\|_{L_T^\infty \dot H^{s_0}} \|u\|_{L_T^\infty \dot H^2} \int_0^T \left(\|u(t')\|_{\dot H^{s_0+1}} + \|u(t')\|_{\dot H_q^{2\sigma-1}} \right)^{p-1} dt' \\ & + 2\|u_t\|_{L_T^\infty \dot H^1} \int_0^T \left(\|u(t')\|_{\dot H^{s_0+1}} + \|u(t')\|_{\dot H_q^{2\sigma-1}} \right)^p dt' \\ & + p \, \|u\|_{L_T^\infty \dot H^2} \int_0^T \|u_t(t')\|_{\dot H_q^{2\sigma-2}} \left(\|u(t')\|_{\dot H^{s_0+1}} + \|u(t')\|_{\dot H_q^{2\sigma-1}} \right)^{p-1} dt'. \end{aligned} \tag{3.17}$$

Hence

$$\begin{aligned} I_4 \lesssim{} & p \, T \|u_t\|_{L_T^\infty \dot H^{s_0}} \|u\|_{L_T^\infty \dot H^2} \|u\|_{L_T^\infty \dot H^{s_0+1}}^{p-1} \\ & + p\|u_t\|_{L_T^\infty \dot H^{s_0}} \|u\|_{L_T^\infty \dot H^2} \int_0^T \|u(t')\|_{\dot H_q^{2\sigma-1}}^{p-1} \, dt' \\ & + p\|u\|_{L_T^\infty \dot H^2} \|u\|_{L_T^\infty \dot H^{s_0+1}}^{p-1} \int_0^T \|u_t(t')\|_{\dot H_q^{2\sigma-2}} \, dt' \\ & + p\|u\|_{L_T^\infty \dot H^2} \int_0^T \|u_t(t')\|_{\dot H_q^{2\sigma-2}} \|u(t')\|_{\dot H_q^{2\sigma-1}}^{p-1} \, dt' \\ & + 2T\|u_t\|_{L_T^\infty \dot H^1} \|u\|_{L_T^\infty \dot H^{s_0+1}}^p + 2\|u_t\|_{L_T^\infty \dot H^1} \int_0^T \|u(t')\|_{\dot H_q^{2\sigma-1}}^p \, dt'. \end{aligned} \tag{3.18}$$

Using the Hölder inequality we have

$$\begin{aligned}
&\int_0^T \|u(t')\|^{p-1}_{\dot H^{2\sigma-1}_q}\,dt' \le T^{1+(1-p)/r}\|u\|^{p-1}_{L^r_T\dot H^{2\sigma-1}_q},\\
&\int_0^T \|u_t(t')\|_{\dot H^{2\sigma-2}_q}\,dt' \le T^{1-1/r}\|u_t\|_{L^r_T\dot H^{2\sigma-2}_q},\\
&\int_0^T \|u_t(t')\|_{\dot H^{2\sigma-2}_q}\|u(t')\|^{p-1}_{\dot H^{2\sigma-1}_q}\,dt' \le T^{1-p/r}\|u_t\|_{L^r_T\dot H^{2\sigma-2}_q}\|u\|^{p-1}_{L^r_T\dot H^{2\sigma-1}_q},\\
&\int_0^T \|u(t')\|^{p}_{\dot H^{2\sigma-1}_q}\,dt' \le T^{1-p/r}\|u\|^{p}_{L^r_T\dot H^{2\sigma-1}_q}.
\end{aligned} \tag{3.19}$$

From (3.16), (3.18) and (3.19) we have

$$\begin{aligned}
|||\mathbb{F}(u)|||_X \le\, & C(1+T)(\|u_0\|_{\dot H^1}+\|u_0\|_{\dot H^2}+\|u_1\|_{H^1})\\
& + CT|m_2|\,M + C|k_p|P(T)M^{p+1}
\end{aligned} \tag{3.20}$$

with $P(T) = T + T^{1-p/r} + T^{1+(1-p)/r} + T^{1-1/r}$.

Let $\delta = \|u_0\|_{\dot H^1} + \|u_0\|_{\dot H^2} + \|u_1\|_{H^1}$, $M = 2C(1+T)\delta$ and T such that

$$C|m_2|T + C|k_p|P(T)M^p \le 1/2 \tag{3.21}$$

then we have that $\mathbb{F}(X^M_T) \subset X^M_T$.

We can choose T and M such that $\mathbb{F}$ will be a contraction because

$$\mathbb{F}(u)(t) - \mathbb{F}(\tilde u)(t) = \int_0^t K(t-t')B^{-1}(G(u)-G(\tilde u))(t')\,dt',$$

from (3.13) and (3.14)

$$|||\mathbb{F}(u)-\mathbb{F}(\tilde u)|||_X \le C\int_0^T \|(-\Delta)^{\frac12}B^{-1}(G(u)-G(\tilde u))(t')\|_2\,dt'.$$

By the inequality in the Remark 3.2.1

$$\begin{aligned}
|||\mathbb{F}(u)-\mathbb{F}(\tilde u)|||_X \le\, & C\,T|m_2|\|u-\tilde u\|_{L^\infty_T\dot H^1}\\
& + C|k_p|\int_0^T \|(F_p(u)-F_p(\tilde u))(t')\|_2\,dt'.
\end{aligned}$$

To estimate the last term we notice that

$$\|\nabla^p u - \nabla^p \tilde{u}\|_\infty \lesssim \|\nabla(u-\tilde{u})\|_\infty \times \\ \{\| \sum_{k=0}^{p-1} (\partial_x u)^{p-1-k} (\partial_x \tilde{u})^k \|_\infty + \| \sum_{k=0}^{p-1} (\partial_y u)^{p-1-k} (\partial_y \tilde{u})^k \|_\infty\} \tag{3.22}$$

and

$$\begin{aligned} &\|\Delta_p u - \Delta_p \tilde{u}\|_2 \\ &\quad \lesssim p\|(\partial_x u)^{p-1} - (\partial_x \tilde{u})^{p-1}\|_\infty \|\partial_x^2 u\|_2 + p\|\partial_x \tilde{u}\|_\infty^{p-1} \|\partial_x^2 (u-\tilde{u})\|_2 \\ &\quad\quad + p\|(\partial_y u)^{p-1} - (\partial_y \tilde{u})^{p-1}\|_\infty \|\partial_y^2 u\|_2 + p\|\partial_y \tilde{u}\|_\infty^{p-1} \|\partial_y^2 (u-\tilde{u})\|_2 \\ &\quad \lesssim p\|\nabla(u-\tilde{u})\|_\infty \| \sum_{k=0}^{p-2} (\partial_x u)^{p-2-k} (\partial_x \tilde{u})^k \|_\infty \|u\|_{\dot{H}^2} \\ &\quad\quad + p\|\nabla(u-\tilde{u})\|_\infty \| \sum_{k=0}^{p-2} (\partial_y u)^{p-2-k} (\partial_y \tilde{u})^k \|_\infty \|u\|_{\dot{H}^2} \\ &\quad\quad + p\|\nabla \tilde{u}\|_\infty^{p-1} \|u-\tilde{u}\|_{\dot{H}^2} . \end{aligned} \tag{3.23}$$

Since

$$\begin{aligned} F_p(u) - F_p(\tilde{u}) &= 2\,\nabla(u-\tilde{u})_t \cdot \nabla^p u + (u-\tilde{u})_t \Delta_p u \\ &\quad + 2\,\nabla \tilde{u}_t \cdot (\nabla^p u - \nabla^p \tilde{u}) + \tilde{u}_t (\Delta_p u - \Delta_p \tilde{u}) \end{aligned}$$

and using the definitions of $\nabla^p u$ and $\Delta_p u$ (see (5) and (6)) we have

$$\begin{aligned} \int_0^T \|(F_p(u) - F_p(\tilde{u}))(t')\|_2\, dt' \lesssim & \int_0^T \|\nabla^p u\|_\infty \|(u-\tilde{u})_t\|_{\dot{H}^1}\, dt' \\ & + p \int_0^T \|(u-\tilde{u})_t\|_\infty \|\nabla u\|_\infty^{p-1} \|u\|_{\dot{H}^2}\, dt' \\ & + \int_0^T \|\nabla^p u - \nabla^p \tilde{u}\|_\infty \|\tilde{u}_t\|_{\dot{H}^1}\, dt' \\ & + \int_0^T \|\tilde{u}_t\|_\infty \|\Delta_p u - \Delta_p \tilde{u}\|_2\, dt'. \end{aligned} \tag{3.24}$$

Remark 3.2.2. *We recall that, by Lemma 1.3.3 and the interpolation result (1.33) we have*

$$\|\nabla w\|_\infty \lesssim \|w\|_{\dot{H}^{s_0+1}} + \|w\|_{\dot{H}_q^{2\sigma-1}} \lesssim \|w\|_{\dot{H}^1}^{1-s_0} \|w\|_{\dot{H}^2}^{s_0} + \|w\|_{\dot{H}_q^{2\sigma-1}}$$

and that

$$\int_0^T \|\nabla w(t',\cdot)\|_\infty \, dt' \lesssim T\|w\|_{L^\infty_T \dot{H}^{s_0+1}} + T^{1/r'}\|w\|_{L^r_T \dot{H}^{2\sigma-1}} \lesssim (T+T^{1/r'})|||w|||_X.$$

To simplify the proof we only consider in the last terms right hand side in (3.24) the most difficult ones. Using the Remark 3.2.2, (3.22) and (3.23) we just need to estimate the following two expressions:

$$\int_0^T \|\tilde{u}_t\|_{\dot{H}^1} \|u-\tilde{u}\|_{\dot{H}^{2\sigma-1}_q} \|u\|^{p-1-k}_{\dot{H}^{2\sigma-1}_q} \|\tilde{u}\|^k_{\dot{H}^{2\sigma-1}_q} \, dt' \tag{3.25}$$

and

$$\int_0^T \|\tilde{u}_t\|_{\dot{H}^1} \|u-\tilde{u}\|_{\dot{H}^{s_0+1}} \|u\|^{p-1-k}_{\dot{H}^{2\sigma-1}_q} \|\tilde{u}\|^k_{\dot{H}^{2\sigma-1}_q} \, dt'. \tag{3.26}$$

Using Hölder's inequality we have

$$\begin{aligned} &\int_0^T \|\tilde{u}_t\|_{\dot{H}^1} \|u-\tilde{u}\|_{\dot{H}^{2\sigma-1}_q} \|u\|^{p-1-k}_{\dot{H}^{2\sigma-1}_q} \|\tilde{u}\|^k_{\dot{H}^{2\sigma-1}_q} \, dt' \\ &\lesssim T^{1-p/r} |||u|||_X^{p-1-k} |||\tilde{u}|||_X^{k+1} |||u-\tilde{u}|||_X \end{aligned} \tag{3.27}$$

and

$$\begin{aligned} &\int_0^T \|\tilde{u}_t\|_{\dot{H}^1} \|u-\tilde{u}\|_{\dot{H}^{s_0+1}} \|u\|^{p-1-k}_{\dot{H}^{2\sigma-1}_q} \|\tilde{u}\|^k_{\dot{H}^{2\sigma-1}_q} \, dt' \\ &\lesssim \|\tilde{u}_t\|_{L^\infty_T H^1} \|u-\tilde{u}\|^{1-s_0}_{L^\infty_T \dot{H}^1} \|u-\tilde{u}\|^{s_0}_{L^\infty_T \dot{H}^2} \int_0^T \|u\|^{p-1-k}_{\dot{H}^{2\sigma-1}_q} \|\tilde{u}\|^k_{\dot{H}^{2\sigma-1}_q} \, dt' \\ &\lesssim T^{1+(1-p)/r} |||u|||_X^{p-1-k} |||\tilde{u}|||_X^{k+1} |||u-\tilde{u}|||_X. \end{aligned} \tag{3.28}$$

A similar argument applied to the remainder terms on the right hand side of (3.24) implies

$$\begin{aligned} &|||\mathbb{F}(u)-\mathbb{F}(\tilde{u})|||_X \\ &\lesssim (T|m_2| + |k_p|(T + T^{1-p/r} + T^{1+(1-p)/r} + T^{1-1/r})M^p)|||u-\tilde{u}|||_X. \end{aligned} \tag{3.29}$$

Hence by standard arguments we can guarantee the existence and uniqueness of a solution of the Cauchy problem (3.5). □

Proof of the Corollary 3.1.3

The result follows from the local theory (Theorem 3.1.1) and *a priori* estimate of $\|u(t)\|_{\dot{H}^1(\mathbb{R}^2)}$, $\|u(t)\|_{\dot{H}^2(\mathbb{R}^2)}$ and $\|u_t(t)\|_{H^1(\mathbb{R}^2)}$ (the Remark 3.1.2).
$\square$

Chapter 4

Local well-posedness of the generalized Benney-Luke equations

4.1 Introduction and notations

In this chapter we will study the IVP associated to an equivalent version of generalized Benney-Luke equation (9)

$$\Phi_{tt} - \Delta\Phi + \mu(a\Delta^2\Phi - b\Delta\Phi_{tt}) + \epsilon(\Phi_t\Delta\Phi + 2\nabla\Phi\cdot\nabla\Phi_t) + f(\nabla\Phi) = 0 \quad (4.1)$$

where $f(\nabla\Phi) = \beta\nabla\cdot(|\nabla\Phi|^m\nabla\Phi)$, $\Phi(t, \mathbf{x})$ is a real valued function, $(t, \mathbf{x}) \in \mathbb{R}_+ \times \mathbb{R}^2$, $\mathbb{R}_+ = [0, \infty)$, a, b, μ, m, and ϵ are positive real constants, β constant. The equation (4.1) with $m = 2$ is a model to describe dispersive and weakly nonlinear long water waves with small amplitude. If we omit the last term on the left hand side of equation (4.1), it becomes the isotropic Benney-Luke equation (1).

Wang, Xu and Chen in [5] studied the Cauchy problem associated to an n-dimensional generalized Benney-Luke equation (4.1) where $n = 1, 2, 3, 4$.

They proved the existence and the uniqueness of the global solution in $H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ for the $\beta \le 0$ case using energy conservation law and the nonexistence of the global solutions of the Cauchy problem for the $\beta > 0$ case.

4.2 Statements of the results

The first result provides local well-possedness of the IVP associated to the equation (4.1) in $H^s(\mathbb{R}^2) \times H^{s-1}(\mathbb{R}^2)$ for $\frac{9}{5} < s \le 2$ and the local regularity results.

Theorem 4.2.1. *Let* $\frac{9}{5} < s \le 2$ *and* $m = \{1, 2, 3, 4\}$, $\epsilon = 0$. *Assume that* $\Phi_0 \in H^s(\mathbb{R}^2)$, $\Phi_1 \in H^{s-1}(\mathbb{R}^2)$. *Then there exist* $T > 0$ *depending on* s *and* $\|\Phi_0\|_{H^s(\mathbb{R}^2)} + \|\Phi_1\|_{H^{s-1}(\mathbb{R}^2)}$ *such that (4.1) has a unique solution* Φ *satisfying* $\Phi(0, \mathbf{x}) = \Phi_0(\mathbf{x})$, $\Phi_t(0, \mathbf{x}) = \Phi_1(\mathbf{x})$,

$$\Phi \in \bigcap_{j=0}^{1} C^j([0, T); H^{s-j}(\mathbb{R}^2))$$

and

$$\left(\int_0^T \|(-\Delta)^{\sigma-1/2}\Phi(t, \cdot)\|_{L^q}^r \, dt\right)^{1/r} < \infty,$$

$$\left(\int_0^T \|(-\Delta)^{\sigma-1}\Phi_t(t, \cdot)\|_{L^q}^r \, dt\right)^{1/r} < \infty,$$

for any $q \in (\frac{2}{4s-7}, \frac{2}{2-s})$, $\sigma = \frac{s}{2} + \frac{1}{8} + \frac{3}{4q}$, *and* $r = \frac{4q}{q-2}$.

In addition, the solution map from the initial data to the solution space is locally Lipschitz.

The main result of local well-posedness for the generalized Benney-Luke equation is in the energy space, $\dot{H}^1(\mathbb{R}^2) \cap \dot{H}^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ is as follows:

Theorem 4.2.2. *Let m be a positive integer and assume that $\Phi_0 \in \dot{H}^1(\mathbb{R}^2)\cap \dot{H}^2(\mathbb{R}^2)$ and $\Phi_1 \in H^1(\mathbb{R}^2)$. Then there exists $T = T(\|\Phi_0\|_{\dot{H}^1}, \|\Phi_0\|_{\dot{H}^2}, \|\Phi_1\|_{H^1})$, $T > 0$, such that (4.1) has a unique solution Φ satisfying $\Phi(0,\mathbf{x}) = \Phi_0(\mathbf{x})$, $\Phi_t(0,\mathbf{x}) = \Phi_1(\mathbf{x})$,*

$$\Phi \in C(0,T;\dot{H}^2(\mathbb{R}^2)\cap \dot{H}^1(\mathbb{R}^2)) \cap L^r(0,T;\dot{H}^{2\sigma-1}_q(\mathbb{R}^2)),$$

$$\Phi_t \in C(0,T;H^1(\mathbb{R}^2)) \cap L^r(0,T;\dot{H}^{2\sigma-2}_q(\mathbb{R}^2)),$$

with $r = \dfrac{4q}{q-2}$, $\sigma = \dfrac{9}{8} + \dfrac{3}{4q}$ and $2 < q < q(m)$,

$$q(m) = \begin{cases} \infty, & m = 1,2,3,4,\\ \dfrac{2m}{m-4}, & m > 4. \end{cases} \tag{4.2}$$

We also have

$$\nabla\Phi, \Phi_t \in C(0,T;H^1(\mathbb{R}^2)) \cap L^4(0,T;L^\infty(\mathbb{R}^2)). \tag{4.3}$$

Moreover, for all $0 < T' < T$ there exists a neighborhood V of $(\Phi_0,\Phi_1) \in \dot{H}^1(\mathbb{R}^2)\cap \dot{H}^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ such that the map data solution

$$\begin{array}{ccc} V & \to & C(0,T';\dot{H}^2(\mathbb{R}^2)\cap\dot{H}^1(\mathbb{R}^2)) \cap L^r(0,T';\dot{H}^{2\sigma-1}_q(\mathbb{R}^2)) \\ (\tilde{\Phi}_0,\tilde{\Phi}_1) & \to & \tilde{\Phi}(t) \end{array}$$

is Lipschitz.

Proposition 4.2.3. *Suposse that $\Phi_0 \in \dot{H}^1(\mathbb{R}^2)\cap\dot{H}^2(\mathbb{R}^2)$, $\Phi_1 \in H^1(\mathbb{R}^2)$, $\Phi(t,\mathbf{x}) \in C(0,T;\dot{H}^2(\mathbb{R}^2)\cap\dot{H}^1(\mathbb{R}^2)) \cap C^1(0,T;H^1(\mathbb{R}^2)) \cap C^2(0,T;L^2(\mathbb{R}^2))$ is the solution for the Cauchy problem associated to (4.1). Then for all $t \in (0,T)$*

$$\begin{aligned} E(t) &= \|\partial_t\Phi(t)\|_2^2 + \mu b\|\partial_t\Phi(t)\|_{\dot{H}^1}^2 \\ &\quad + \|\Phi(t)\|_{\dot{H}^1}^2 + \mu a\|\Phi(t)\|_{\dot{H}^2}^2 - \frac{2\beta}{m+2}\|\nabla\Phi(t)\|_{L^{m+2}}^{m+2} = E(0) \end{aligned} \tag{4.4}$$

Proof. See ([5]).

Using the previous proposition, it is possible to establish an *a priori* estimate to prove the following global result for the generalized Benney-Luke equation (4.1) in the energy space.

Theorem 4.2.4. *Let $m \geq 1$ integer and $\beta < 0$. For any $T > 0$, $\Phi_0 \in \dot{H}^1(\mathbb{R}^2) \cap \dot{H}^2(\mathbb{R}^2)$ and $\Phi_1 \in H^1(\mathbb{R}^2)$ there exists a unique solution Φ of (4.1) such that $\Phi(0, \mathbf{x}) = \Phi_0(\mathbf{x})$, $\Phi_t(0, \mathbf{x}) = \Phi_1(\mathbf{x})$ and*

$$\nabla \Phi \in C(0, T; H^1(\mathbb{R}^2)), \;\; \partial_t \Phi \in C(0, T; H^1(\mathbb{R}^2)).$$

In following result we establish that the local solution of the Cauchy problem associated to the generalized Benney-Luke equation (4.1) possesses more local regularity than the initial data, for instance $\nabla\Phi(t, \cdot), \Phi_t(t, \cdot) \in L^\infty(\mathbb{R}^2)$ a.e. $t \in (0, T]$ when the initial data $\Phi_0 \in H^2(\mathbb{R}^2)$ and $\Phi_1 \in H^1(\mathbb{R}^2)$.

Theorem 4.2.5 (Local well-posedness and local regularity)**.** *Let $m \geq 1$ and assume that $\Phi_0 \in H^2(\mathbb{R}^2)$ and $\Phi_1 \in H^1(\mathbb{R}^2)$. Then there exists $T > 0$, $T = T(\|\Phi_0\|_{H^2(\mathbb{R}^2)}, \|\Phi_1\|_{H^1(\mathbb{R}^2)})$ such that (4.1) has a unique solution Φ satisfying $\Phi(0, \mathbf{x}) = \Phi_0(\mathbf{x}), \; \Phi_t(0, \mathbf{x}) = \Phi_1(\mathbf{x})$*

$$\Phi \in C(0, T; H^2(\mathbb{R}^2)), \Phi_t \in C(0, T; H^1(\mathbb{R}^2)).$$

In addition,

$$\left(\int_0^T \|(-\Delta)^{\sigma - 1/2} \Phi(t, \cdot)\|_{L^q}^r \, dt \right)^{1/r} < \infty,$$

$$\left(\int_0^T \|(-\Delta)^{\sigma - 1} \Phi_t(t, \cdot)\|_{L^q}^r \, dt \right)^{1/r} < \infty,$$

with $r = \dfrac{4q}{q-2}$, $\sigma = \dfrac{9}{8} + \dfrac{3}{4q}$ and $2 < q < \infty$.

In particular,

$$\int_0^T \|(\nabla\Phi, \Phi_t)(t)\|_\infty^4 \, dt < \infty.$$

Moreover, for all $0 < T' < T$ *there exists a neighborhood* V *of* $(\Phi_0, \Phi_1) \in H^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ *such that the map data solution*

$$\begin{array}{ccc} V & \to & C(0,T'; H^2(\mathbb{R}^2)) \cap L^r(0,T'; \dot{H}_q^{2\sigma-1}(\mathbb{R}^2)) \\ (\tilde{\Phi}_0, \tilde{\Phi}_1) & \to & \tilde{\Phi}(t) \end{array}$$

is Lipschitz.

It is also possible to establish local regularity results for solutions of the generalized Benney-Luke equation in the 3-dimensional case. For all $\Phi_0 \in H^s(\mathbb{R}^3)$ and $\Phi_1 \in H^{s-1}(\mathbb{R}^3)$ with $2 < s \leqslant 5/2$ exist a unique solution $\Phi(t, \cdot)$ such that $\nabla\Phi(t), \Phi_t(t) \in L^\infty(\mathbb{R}^3)$ a.e. $t \in (0, T)$.

Theorem 4.2.6 (Local well-posedness and local regularity)**.** *Assume that* $(u_0, u_1) \in H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$ *and* $2 < s \leqslant 5/2$*. Then there exists* $T > 0$ *such that the Cauchy problem:*

$$\begin{cases} (1 - bc^{-2}\Delta)(u_{tt} - \Delta u) = c^{-2}\left((1 - c^2)\Delta u - u_t \Delta u - 2\nabla u \cdot \nabla u_t\right) - \alpha f(\nabla u), \\ u(0, \mathbf{x}) = u_0(\mathbf{x}), \quad u_t(0, \mathbf{x}) = u_1(\mathbf{x}), \end{cases} \tag{4.5}$$

has a unique solution u *satisfying*

$$u \in C(0, T; H^s(\mathbb{R}^3)), \partial_t u \in C(0, T; H^{s-1}(\mathbb{R}^3)),$$

in addition

$$\left(\int_0^T \|(-\Delta)^{\sigma - 1/2} u(t, \cdot)\|_{L^q}^r \, dt\right)^{1/r} < \infty,$$

$$\left(\int_0^T \|(-\Delta)^{\sigma - 1} u_t(t, \cdot)\|_{L^q}^r \, dt\right)^{1/r} < \infty,$$

where $r = \dfrac{2q}{q-2}$, $\sigma = \dfrac{s}{2} + \dfrac{1}{q}$ *and* $(s-2)^{-1} < q < \infty$.

And

$$\int_0^T \|(\nabla u, u_t)(t)\|_\infty^2 \, dt < \infty. \tag{4.6}$$

With $f(\nabla u) = \nabla \cdot (|\nabla u|^m \nabla u)$ *and* u *is such that*

$$\Phi(t, \mathbf{x}) := \frac{\sqrt{\mu}}{\epsilon} u(\frac{t}{\sqrt{\mu}}, \frac{c}{\sqrt{\mu}} \ \mathbf{x}), \tag{4.7}$$

Φ *satisfying the generalized Benney-Luke equation (4.1),* $c^2 = a/b$*,* $(t, \mathbf{x}) \in \mathbb{R}_+ \times \mathbb{R}^3$*,* $\mathbb{R}_+ = [0, \infty)$*,* a *and* b *are positive real constants.*

Using (4.7) with Φ satisfying (4.1) in $\mathbb{R}^2$, we have that the initial value problem associated to the generalized Benney-Luke equation (4.1) in $\mathbb{R}^2$, is equivalent to

$$\begin{cases} (1 - b\Delta)(u_{tt} - c^2 \Delta u) = (1 - c^2)\Delta u - H(u) \\ u(0, \mathbf{x}) = u_0(\mathbf{x}) \quad u_t(0, \mathbf{x}) = u_1(\mathbf{x}) \end{cases} \tag{4.8}$$

where

$$H(u) = u_t \Delta u + 2\nabla u \cdot \nabla u_t + \beta_m \nabla \cdot (|\nabla u|^m \nabla u),$$

$c^2 = \dfrac{a}{b}$, $u_i(\mathbf{x}) = \dfrac{\epsilon}{\sqrt{\mu}} \Phi_i(\sqrt{\mu}\,\mathbf{x})$, $i = 0, 1$, $\mathbf{x} \in \mathbb{R}^2$ and $\beta_m = \frac{\beta}{\epsilon^m}$.

Considering that the IVP of the generalized Benney-Luke equations (4.1) and the Cauchy problem associated (4.8) are equivalent, we use the Strichartz estimates of the wave equations to prove our results of local well-posedness and local regularity.

4.3 Proof of Theorem 4.2.1

We begin by rewriting our initial value problem (4.8) in the equivalent form

$$(1 - bc^{-2}\Delta)(u_{tt} - \Delta u) = c^{-2}(1 - c^2)\Delta u - c^2 \beta_m F_m(u, \nabla u), \tag{4.9}$$

$$u(0) = u_0, \qquad u_t(0) = u_1, \tag{4.10}$$

where

$$F_m(u, \nabla u) = \nabla \cdot (|\nabla u|^m \nabla u). \tag{4.11}$$

Fix $9/5 < s \leq 2$ and take $q \in (\frac{2}{4s-7}, \frac{2}{2-s})$ and for $T, M > 0$, we define the complete metric space

$$X_T^M = \{u \in ([0, T); H^s(\mathbb{R}^2) : |||u||| \leq M\},$$

where

$$|||u||| = \|u\|_{L_T^\infty H^s(\mathbb{R}^2)} + (\int_0^T \|(-\Delta)^{\sigma-1/2} u(t, \cdot)\|_{L^q}^r \, dt)^{1/r},$$

with $r = \frac{4q}{q-2}$ and $\sigma = \frac{s}{2} + \frac{1}{8} + \frac{3}{4q}$.

We will show that for an appropriate choice of T and M the operator

$$\mathbb{F}(u)(t) = \dot{\mathbf{K}}(t)u_0 + \mathbf{K}(t)u_1 + \int_0^t \mathbf{K}(t - t')G(u)(t')\, dt', \tag{4.12}$$

is a contraction of X_T^M into itself.

We will use the notation $G(u) = G_1(u) + G_3(u)$ where

$$G_1 = c^{-2}(1 - c^2)\Delta(1 - bc^{-2}\Delta)^{-1}u \tag{4.13}$$

and

$$G_3 = -c^{-2}\beta_m(1 - bc^{-2}\Delta)^{-1}F_m(u, \nabla u). \tag{4.14}$$

We estimate $\|\mathbb{F}(u)\|_{L_T^\infty H^s}$ using the linear estimate (1.6) and $\|\mathbb{F}(u)\|_{L_T^r \dot{H}^{2\sigma-1}}$ using the Strichartz estimates, Proposition 1.2.1, as follows,

$$\begin{aligned} \|\mathbb{F}(u)\|_{L_T^\infty H^s} + \|\mathbb{F}(u)\|_{L_T^r \dot{H}_q^{2\sigma-1}} &\lesssim \|u_0\|_{H^s} + (1+T)\|u_1\|_{H^{s-1}} \\ &+ \int_0^T (T - t')\|G(u)(t')\|_2 \, dt' + \int_0^T \|(-\Delta)^{(s-1)/2} G(u)(t')\|_2 \, dt'. \end{aligned} \tag{4.15}$$

To estimate the fractional derivative of the non-linear term, we use that for $1 \le s \le 2$

$$\begin{aligned}\|(-\Delta)^{(s-1)/2} G_3(u)(t')\|_2^2 &\lesssim \||\nabla u|^m u_x\|_2 + \||\nabla u|^m u_y\|_2^2 \\ &\lesssim \|\nabla u\|_\infty^{2m} \{\|u_x\|_2^2 + \|u_y\|_2^2\} \\ &\lesssim \|\nabla u\|_\infty^{2m} \|\nabla u\|_2^2\end{aligned} \tag{4.16}$$

and

$$\|(-\Delta)^{(s-1)/2} G_1(u)(t')\|_2 \lesssim \|(-\Delta)^{(s-1)/2} u(t')\|_2 \lesssim \|u(t')\|_{H^s}, \tag{4.17}$$

for all s.

Using Lemma 1.3.5 we have

$$\|\nabla u\|_\infty^m \lesssim (\|\nabla u\|_{\dot{H}^{s_0}} + \|\nabla u\|_{\dot{H}_q^{2\sigma-2}})^m \quad \lesssim \|u\|_{\dot{H}^{s_0+1}}^m + \|u\|_{\dot{H}_q^{2\sigma-1}}^m. \tag{4.18}$$

Therefore, if $s_0 + 1 < s$, we have

$$\|\nabla u\|_\infty^m \lesssim \|u\|_{H^s}^m + \|u\|_{\dot{H}_q^{2\sigma-1}}^m.$$

Since

$$\begin{aligned}\|(-\Delta)^{(s-1)/2} G_3(u)\|_2 &\lesssim \|u\|_{H^s} (\|u\|_{H^s}^m + \|u\|_{\dot{H}_q^{2\sigma-1}}^m) \\ &\lesssim \|u\|_{H^s}^{m+1} + \|u\|_{H^s} \|u\|_{\dot{H}_q^{2\sigma-1}}^m,\end{aligned} \tag{4.19}$$

then

$$\begin{aligned}\int_0^T \|(-\Delta)^{(s-1)/2} G_3(u(t'))\|_2 \, dt' &\lesssim \int_0^T \|u(t')\|_{H^s}^{m+1} \, dt' \\ &\quad + \int_0^T \|u(t')\|_{H^s} \|u(t')\|_{\dot{H}_q^{2\sigma-1}}^m \, dt'.\end{aligned} \tag{4.20}$$

Therefore

$$\begin{aligned}\int_0^T \|(-\Delta)^{(s-1)/2} G_3(u(t'))\|_2 \, dt' &\lesssim T(\|u\|_{L_T^\infty H^s})^{m+1} \\ &\quad + \|u\|_{L_T^\infty H^s} \int_0^T \|u(t')\|_{\dot{H}_q^{2\sigma-1}}^m \, dt'.\end{aligned} \tag{4.21}$$

Using the Hölder's inequality we get

$$\int_0^T \|u(t')\|^m_{\dot{H}^{2\sigma-1}_q}\,dt' \lesssim T^{1-m/r}\|u\|^m_{L^r_T\dot{H}^{2\sigma-1}_q}. \tag{4.22}$$

Combining (4.16), (4.18) and the Hölder's inequality we conclude

$$\begin{aligned}\int_0^T (T-t')\|G_3(u)(t')\|_2\,dt' \lesssim\, & T^2(\|u\|_{L^\infty_T H^s})^{m+1} \\ & + T^{2-m/r}\|u\|_{L^\infty_T H^s}\|u\|^m_{L^r_T\dot{H}^{2\sigma-1}_q}.\end{aligned} \tag{4.23}$$

□

4.4 Proof of Theorem 4.2.2

First, rewriting the IVP associated to equation (4.1) in the equivalent form (4.8), we will use the Strichartz estimates for the wave equation, Lemma 1.3.9 and the fixed point theorem.

Fix q, $2 < q < q(m)$. Let $u_0 \in \dot{H}^1(\mathbb{R}^2) \cap \dot{H}^2(\mathbb{R}^2)$, $u_1 \in H^1(\mathbb{R}^2)$ and

$$\mathbb{F}_{(u_0,u_1)}(u)(t) = \mathbb{F}(u)(t) = \dot{\mathbf{K}}(t)u_0 + \mathbf{K}(t)u_1 + \int_0^t \mathbf{K}(t-t')G(u)(t')\,dt', \tag{4.24}$$

with $G(u) = G_1(u) + G_2(u) + G_3(u)$ where

$$G_1 = c^{-2}(1-c^2)\Delta(1-bc^{-2}\Delta)^{-1}u, \tag{4.25}$$

$$G_2(u) = -c^{-2}(1-bc^{-2}\Delta)^{-1}[\Delta, u]u_t. \tag{4.26}$$

and

$$G_3 = -c^{-2}\beta_m(1-bc^{-2}\Delta)^{-1}F_m(u,\nabla u). \tag{4.27}$$

We define the complete metric space

$$Y_T^M = \{u \in C([0,T];\dot{H}^2(\mathbb{R}^2)\cap\dot{H}^1(\mathbb{R}^2)) : |||u|||_Y \le M\}$$

with

$$
\begin{aligned}
|||u|||_Y &= \|u\|_{L_T^\infty \dot{H}^2} + \|u\|_{L_T^\infty \dot{H}^1} + \|u_t\|_{L_T^\infty H^1} \\
&+ \left(\int_0^T \|(-\Delta)^{\sigma-1/2} u(t)\|_q^r \, dt \right)^{1/r} + \left(\int_0^T \|(-\Delta)^{\sigma-1} u_t(t)\|_q^r \, dt \right)^{1/r},
\end{aligned} \tag{4.28}
$$

$r = \dfrac{4q}{q-2}$, $\sigma = \dfrac{9}{8} + \dfrac{3}{4q}$.

We shall prove that for an appropriate choice of T and M the operator given by (4.24) is a contraction on Y_T^M.

Let $r_1 = r,\ q_1 = q,\ r_2 = \infty,\ q_2 = 2,\ \rho_1 = 2\sigma - 1,\ \mu = 2$ and $-\rho_2 = 1$, then $r_i,\ q_i,\ \rho_i$ and μ, $i = 1, 2$, satisfy (1.17), (1.18), (1.19), (1.20). Now, using the linear estimate (1.6), the Strichartz estimates (1.21) and (1.22) with $r_i,\ q_i,\ \rho_i$ and μ, $i = 1, 2$ we get

$$
\begin{aligned}
|||\mathbb{F}(u)|||_Y \lesssim & \|u_0\|_{\dot{H}^2} + \|u_0\|_{\dot{H}^1} \\
& + (1+T)\|u_1\|_{H^1} + \int_0^T \|(-\Delta)^{\frac{1}{2}} G(u)(t')\|_{L^2} \, dt'.
\end{aligned} \tag{4.29}
$$

Using the inequalities (2.17) and Lemma 1.3.9 we get

$$
\begin{aligned}
|||\mathbb{F}(u)|||_Y \le & C \left(\|u_0\|_{\dot{H}^2} + \|u_0\|_{\dot{H}^1} + (1+T)\|u_1\|_{H^1} \right) \\
& + C \left(|1 - c^2| T \ + (T^{\beta-1} + T)|||u|||_Y \right) |||u|||_Y \\
& + C(T + T^{1-m/r})|||u|||_Y^{m+1},
\end{aligned} \tag{4.30}
$$

where $\beta = 1 + \dfrac{1}{r'}$.

Let $\delta = \|u_0\|_{\dot{H}^2} + \|u_0\|_{\dot{H}^1} + \|u_1\|_{H^1}$, $M = 2C(1+T)\delta$ and T such that

$$
C|1 - c^2|T + C(T + T^{\beta-1})M + C(T + T^{1-m/r})M^m \le 1/2 \tag{4.31}
$$

then we have that $\mathbb{F}(Y_T^M) \subset Y_T^M$.

In a similar form, it possible to prove that under the same restrictions on M and T, the operator $\mathbb{F}$ is a contraction on Y_T^M. Thus there exists a unique fixed point of $\mathbb{F}$ which is a solution the IVP (4.8). $\square$

4.5 Proof of Theorem 4.2.5

We can write the solution of the IVP associated to (4.1) as

$$u(t) = \dot{\mathbf{K}}(t)u_0 + K(t)u_1 + \int_0^t \mathbf{K}(t-t')G(u)(t')\,dt'. \tag{4.32}$$

with $G(u) = G_1(u) + G_2(u) + G_3(u)$ where

$$G_1 = c^{-2}(1-c^2)\Delta(1-bc^{-2}\Delta)^{-1}u, \tag{4.33}$$

$$G_2(u) = -c^{-2}(1-bc^{-2}\Delta)^{-1}[\Delta, u]u_t. \tag{4.34}$$

and

$$G_3 = -c^{-2}\beta_m(1-bc^{-2}\Delta)^{-1}F_m(u,\nabla u). \tag{4.35}$$

For $M, T > 0$ and $2 < q < q(m)$, define the complete metric space

$$X_T^M = \{u \in C([0,T]; H^2(\mathbb{R}^2)) : |||u||| \leq M\}$$

where

$$|||u||| = \|u\|_{L_T^\infty H^2} + \|u_t\|_{L_T^\infty H^1} + \|u\|_{L_T^r H_q^{2\sigma-1}} + \|u_t\|_{L_T^r H_q^{2\sigma-2}}, \tag{4.36}$$

with $r = \dfrac{4q}{q-2}$ and $\sigma = \dfrac{9}{8} + \dfrac{3}{4q}$.

We shall prove that for an appropriate choice of T and M the operator

$$\mathbb{F}(u)(t) = \dot{\mathbf{K}}(t)u_0 + K(t)u_1 + \int_0^t \mathbf{K}(t-t')G(u)(t')\,dt' \tag{4.37}$$

is a contraction on X_T^M.

We estimate $|||\mathbb{F}(u)|||$ using the linear estimate (1.6) and Stricharzt estimates, as follows,

$$\begin{aligned} |||\mathbb{F}(u)(t)||| \lesssim\ & \|u_0\|_{H^2} + (1+T)\|u_1\|_{H^1} \\ & + \int_0^T (T-t')\|G(u)(t')\|_2\,dt' + \int_0^T \|(-\Delta)^{\frac{1}{2}}G(u)(t')\|_2\,dt'. \end{aligned} \tag{4.38}$$

Using the inequalities (2.16), (2.17) and Lemma 1.3.9 we get

$$
\begin{aligned}
|||\mathbb{F}(u)||| \leq & C\|u_0\|_{H^2} + C(1+T)\|u_1\|_{H^1} \\
& + C\left(|1-c^2|T^2 + P(T)|||u|||\right)|||u||| \\
& + C\left(T^2 + T + T^{1-m/r}\right)|||u|||^{m+1}
\end{aligned} \tag{4.39}
$$

where $P(T) = T^{\beta} + T^2 + T^{\beta-1} + T$ and $\beta = 1 + \dfrac{1}{r'}$.

Let $\delta = \|u_0\|_{H^2} + \|u_1\|_{H^1}$, $M = 2C(1+T)\delta$ and T such that

$$
C\,|1-c^2|\,(T+T^2) + C\,P(T)\,M + C\left(T^2 + T + T^{1-m/r}\right)M^m \leq 1/2 \tag{4.40}
$$

then we have that $\mathbb{F}(X_T^M) \subset X_T^M$.

In a similar form, it possible to prove that under the same restrictions on M and T, the operator $\mathbb{F}$ is a contraction on X_T^M. Thus there exists a unique fixed point of $\mathbb{F}$ which is a solution the IVP (4.8). $\square$

Chapter 5

Local well-posedness of the generalized Benney-Luke equations in a periodic setting

5.1 Statement and the proof of the result

We are interested in the Cauchy problem for a rescaled version of generalized Benney-Luke equation (gBL)

$$u_{tt} - \Delta u + a\Delta^2 u - b\Delta u_{tt} + 2\nabla u \cdot \nabla u_t + u_t \Delta u + \beta \nabla \cdot (|\nabla u|^m \nabla u) = 0 \quad (5.1)$$

$$u(0, \mathbf{x}) = u_0(\mathbf{x}), \qquad u_t(0, \mathbf{x}) = u_1(\mathbf{x}). \quad (5.2)$$

where $\mathbf{x} = (x, y) \in \mathbb{T} \times \mathbb{R}$ (or $\mathbb{T} \times \mathbb{T}$), and $t \in \mathbb{R}^+$, $a, b > 0$, β constant, and $m > 0$ integer.

The equation (5.1) with $m = 2$ and $a \neq b$ is a model to describe dispersive and weakly non linear long water waves with small amplitude. If we omit the last term of equation (5.1), it becomes a version of the isotropic Benney-Luke equation (1) above mentioned in the periodic setting.

In Theorem 5.1.1 we establish result of local well-posedness in $H^s(\mathbb{T} \times \mathbb{R}) \times H^{s-1}(\mathbb{T} \times \mathbb{R})$ for $2 < s \leq 3$ and $\beta = 0$. It is possible to prove that the associated Cauchy problem to (5.1) is locally well-posed in $H^s(\mathbb{T}\times\mathbb{T})\times H^{s-1}(\mathbb{T}\times\mathbb{T})$ for $2 < s \leq 3$ and $\beta \in \mathbb{R}$, in this case it is necessary to take the Kato-Ponce commutator estimate for functions in $\mathbb{T}\times\mathbb{T}$ (see [12, Lemma A.2 in the Appendix]).

Theorem 5.1.1. *Let $2 < s \leq 3$ and $\beta = 0$. Assume that $u_0 \in H^s(\mathbb{T}\times\mathbb{R})$, $u_1 \in H^{s-1}(\mathbb{T}\times\mathbb{R})$. Then there exist $T > 0$ depending on s and $(\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}})$ such that (5.1) and (5.2) has a unique solution u satisfying*

$$u \in \bigcap_{j=0}^{1} C^j([0,T); H^{s-j}(\mathbb{T}\times\mathbb{R})).$$

In addition, the solution map from the initial data to the solution space is locally Lipschitz.

Proof.

We begin by rewritting the associated Cauchy problem equation (5.1) in the equivalent form

$$\begin{cases} u_{tt} - \Delta u = G(u) \quad on \quad \mathbb{R}\times\mathbb{T}\times\mathbb{R} \\ u(0,.) = u_0(.) \quad u_t(0,.) = u_1(.) \quad on \quad \mathbb{T}\times\mathbb{R} \end{cases} \tag{5.3}$$

where

$$G(u)(\tau) := G_1(u)(\tau) + G_2(u)(\tau)$$

and

$$G_1(u)(\tau) := c^{-2}(1-c^2)\Delta(1-bc^{-2}\Delta)^{-1}u,$$

$$G_2(u)(\tau) := -c^{-2}(1-bc^{-2}\Delta)^{-1}[\Delta, u]u_t,$$

with $c^2 = a/b$.

We will use the fixed point theorem, thus we show first that the following operator

$$\mathbb{F}(u)(t) = \dot{\mathbf{K}}(t)u_0 + \mathbf{K}(t)u_1 + \int_0^t \mathbf{K}(t-t')G(u)(t')\,dt', \tag{5.4}$$

is a contraction on the complete metric space

$$X_T^M = \{u \in C([0,T]; H^s(\mathbb{T}\times\mathbb{R})) : |||u||| \leq M\},$$

where

$$|||u||| := \sup_{[0,T]}\{\|u(t,\cdot)\|_{H^s(\mathbb{T}\times\mathbb{R})} + \|u_t(t,\cdot)\|_{H^{s-1}(\mathbb{T}\times\mathbb{R})}\}$$

for an appropriate choice of T and M.

From now on, we will denote $\|\cdot\|_{L^2(\mathbb{T}\times\mathbb{R})}$ and $\|\cdot\|_{L^\infty(\mathbb{T}\times\mathbb{R})}$ for $\|.\|_2$ and $\|.\|_\infty$, respectively.

We estimate $\|\mathbb{F}(u)(t)\|_{H^s}$ and $\|\partial_t\mathbb{F}(u)(t)\|_{H^s}$ using the linear estimates of wave operator, as follows,

$$\begin{aligned}\|\mathbb{F}(u)(t)\|_{H^s} \lesssim \|u_0\|_{H^s} + (1+t)\|u_1\|_{H^{s-1}} + \int_0^t (t-t')\|G(u)(t')\|_2\,dt' \\ + \int_0^t \|(-\Delta)^{(s-1)/2}G(u)(t')\|_2\,dt',\end{aligned} \tag{5.5}$$

and

$$\begin{aligned}\|\partial_t\mathbb{F}(u)(t)\|_{H^{s-1}} \lesssim \|u_0\|_{H^s} + (1+t)\|u_1\|_{H^{s-1}} + \int_0^t (t-t')\|G(u)(t')\|_2\,dt' \\ + \int_0^t \|(-\Delta)^{(s-1)/2}G(u)(t')\|_2\,dt'.\end{aligned} \tag{5.6}$$

Now we have the following

$$\begin{aligned}|||\mathbb{F}(u)||| \lesssim \|u_0\|_{H^s} + (1+T)\|u_1\|_{H^{s-1}} + \int_0^T (T-t')\|G(u)(t')\|_2\,dt' \\ + \int_0^T \|(-\Delta)^{(s-1)/2}G(u)(t')\|_2\,dt'.\end{aligned} \tag{5.7}$$

It is important to recall that

$$G(u)(\tau) := G_1(u)(\tau) + G_2(u)(\tau)$$

where

$$G_1(u)(\tau) := c^{-2}(1-c^2)\Delta(1-bc^{-2}\Delta)^{-1}u$$

$$G_2(u)(\tau) := -c^{-2}(1-bc^{-2}\Delta)^{-1}[\Delta, u]u_t$$

and note that if $1 \le s \le 3$ then

$$\|(-\Delta)^{(s-1)/2}G_1(u)(\tau)\|_2 \lesssim \|(-\Delta)^{(s-1)/2}u(\tau)\|_2 \tag{5.8}$$

and

$$\|(-\Delta)^{(s-1)/2}G_2(u)(\tau)\|_2 \lesssim \|[\Delta, u]u_t(\tau)\|_2. \tag{5.9}$$

Then

$$\begin{aligned}
\tilde{I}_1 &:= \int_0^T (T-t')\|G(u)(t')\|_2\, dt' \\
&\lesssim \int_0^T (T-t')\{\|u(t')\|_2 + \|[\Delta, u]u_t(t')\|_2\}\, dt' \\
&\lesssim T^2 \sup_{[0,T]} \|u(t)\|_2 + \int_0^T (T-t')\|[\Delta, u]u_t(t')\|_2\, dt'.
\end{aligned} \tag{5.10}$$

Using once again that

$$[\Delta, u]u_t(\tau) = u_t\Delta u(\tau) + 2(\nabla u_t.\nabla u)(\tau) = \Delta(uu_t) - u\Delta u_t$$

it follows that

$$\begin{aligned}
I_1 &:= \int_0^T (T-t')\|[\Delta, u]u_t(t')\|_2\, dt' \\
&\lesssim \int_0^T (T-t')\{\|u_t(t')\|_\infty\|u(t')\|_{H^s} + 2\|\nabla u_t(t')\|_2\|\nabla u(t')\|_\infty\}\, dt'
\end{aligned} \tag{5.11}$$

with $s \ge 2$.

For the Sobolev embedding theorem (see [1], [6], [34]) with $s_0 > 1$, we have the following:

$$\begin{aligned} I_1 &\lesssim \int_0^T (T-t')\{\|u_t(t')\|_{H^{s_0}}\|u(t')\|_{H^s} + 2\|u_t(t')\|_{\dot H^1}\|u(t')\|_{H^{s_0+1}}\}\,dt' \\ &\lesssim T^2\{\|u_t\|_{L_T^\infty H^{s_0}}\|u\|_{L_T^\infty H^s} + 2\|u_t\|_{L_T^\infty \dot H^1}\|u\|_{L_T^\infty H^{s_0+1}}. \end{aligned} \tag{5.12}$$

Moreover, denoting by

$$\tilde I_2 := \int_0^T \|(-\Delta)^{(s-1)/2}G(u)(t')\|_2\,dt' \tag{5.13}$$

and using (5.8) and (5.9) we have that

$$\begin{aligned} \tilde I_2 &\lesssim \int_0^T \|(-\Delta)^{(s-1)/2}u(t')\|_2\,dt' + \int_0^T \|[\Delta, u]u_t(t')\|_2\,dt' \\ &\lesssim T\|u\|_{L_T^\infty \dot H^{s-1}} + \int_0^T \|u_t\|_\infty\|u(t')\|_{H^s}\,dt' \\ &\quad + 2\int_0^T \|\nabla u_t(t')\|_2\|\nabla u(t')\|_\infty\,dt' \end{aligned} \tag{5.14}$$

$$\begin{aligned} \tilde I_2 &\lesssim T\,\|u\|_{L_T^\infty \dot H^{s-1}} + \|u\|_{L_T^\infty \dot H^s}\int_0^T \|u_t(t')\|_{H^{s_0}}\,dt' \\ &\quad + 2\,\|u_t\|_{L_T^\infty \dot H^1}\int_0^T \|u(t')\|_{H^{s_0+1}}\,dt'. \end{aligned} \tag{5.15}$$

Thus

$$\tilde I_2 \lesssim T\,\|u\|_{L_T^\infty \dot H^{s-1}} + T\|u\|_{L_T^\infty \dot H^s}\|u_t\|_{L_T^\infty H^{s_0}} + 2T\|u_t\|_{L_T^\infty \dot H^1}\|u\|_{L_T^\infty H^{s_0+1}}, \tag{5.16}$$

where $s = s_0 + 1 > 2$.

Using all the above estimates we have for $2 < s \le 3$ that

$$\begin{aligned} |||\mathbb{F}(u)(t)||| \lesssim\,& (1+T)\{\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}}\} \\ &+ T^2\|u\|_{L_T^\infty L^2}|1-c^2| + T^2\|u_t\|_{L_T^\infty H^{s-1}}\|u\|_{L_T^\infty H^s} \\ &+ T^2\|u_t\|_{L_T^\infty \dot H^1}\|u\|_{L_T^\infty H^s} \\ &+ T\|u\|_{L_T^\infty \dot H^{s-1}}|1-c^2| + T\|u\|_{L_T^\infty H^s}\|u_t\|_{L_T^\infty H^{s-1}} \\ &+ T\|u_t\|_{L_T^\infty \dot H^1}\|u\|_{L_T^\infty H^s}. \end{aligned} \tag{5.17}$$

Let $\delta = \|u_0\|_{H^s} + \|u_1\|_{H^{s-1}}$, $M = 2\tilde{C}(1+T)\delta$ and T such that

$$\tilde{C}|1-c^2|(T+T^2) + \tilde{C}(T+T^2)M \leq 1/2 \tag{5.18}$$

then from (5.17) we deduce that

$$\mathbb{F}(X_T^M) \subset X_T^M.$$

To prove that $\mathbb{F} : X_T^M \to X_T^M$ is a contraction we will use that

$$G_2(u) - G_2(\tilde{u}) = -c^{-2}(1-bc^{-2}\Delta)^{-1}\{[\Delta,(u-\tilde{u})]u_t + [\Delta,\tilde{u}](u-\tilde{u})_t\}$$

First we observe that

$$\mathbb{F}(u)(t) - \mathbb{F}(\tilde{u})(t) = \int_0^t \mathbf{K}(t-t')(G(u)-G(\tilde{u}))\,dt'$$

then

$$\begin{aligned} |||\mathbb{F}(u) - \mathbb{F}(\tilde{u})||| \lesssim &\int_0^T (T-t')\|(G(u)-G(\tilde{u}))(t')\|_2\,dt' \\ &+ \int_0^T \|(-\Delta)^{(s-1)/2}(G(u)-G(\tilde{u}))(t')\|_2\,dt' \end{aligned} \tag{5.19}$$

and using

$$G(u)(\tau) := G_1(u)(\tau) + G_2(u)(\tau),$$

where

$$G_1(u)(\tau) := c^{-2}(1-c^2)\Delta(1-bc^{-2}\Delta)^{-1}u$$

we see that

$$\begin{aligned} |||\mathbb{F}(u) - \mathbb{F}(\tilde{u})||| \lesssim &\int_0^T (T-t')\|(G_1(u)-G_1(\tilde{u}))(t')\|_2\,dt' \\ &+ \int_0^T \|(-\Delta)^{(s-1)/2}(G_1(u)-G_1(\tilde{u}))(t')\|_2\,dt' \\ &+ \int_0^T (T-t')\|(G_2(u)-G_2(\tilde{u}))(t')\|_2\,dt' \\ &+ \int_0^T \|(-\Delta)^{(s-1)/2}(G_2(u)-G_2(\tilde{u}))(t')\|_2\,dt'. \end{aligned} \tag{5.20}$$

Applying (5.9) it follows that

$$|||\mathbb{F}(u)-\mathbb{F}(\tilde{u})||| \lesssim \int_0^T (1+T-t')\|[\Delta,(u-\tilde{u})]u_t+[\Delta,\tilde{u}](u-\tilde{u})_t\|_2\,dt' + \int_0^T (T-t')\|(G_1(u-\tilde{u}))(t')\|_2\,dt' + \int_0^T \|(-\Delta)^{(s-1)/2}(G_1(u-\tilde{u}))(t')\|_2\,dt'.. \tag{5.21}$$

Using Proposition 1.3.1 and Sobolev embedding theorem we have that for $s>2$

$$\begin{aligned}|||\mathbb{F}(u)-\mathbb{F}(\tilde{u})||| \lesssim& |1-c^2|\int_0^T (T-t')\|(u-\tilde{u})(t')\|_2\,dt' \\ &+|1-c^2|\int_0^T \|(-\Delta)^{(s-1)/2}(u-\tilde{u})(t')\|_2\,dt' \\ &+\int_0^T (1+T-t')\|\nabla(u-\tilde{u})(t')\|_\infty\,\|u_t(t')\|_{\dot{H}^1}\,dt' \\ &+\int_0^T (1+T-t')\|u_t(t')\|_\infty\|(u-\tilde{u})(t')\|_{\dot{H}^2}\,dt' \\ &+\int_0^T (1+T-t')\|\nabla\tilde{u}(t')\|_\infty\|(u-\tilde{u})_t(t')\|_{\dot{H}^1}\,dt' \\ &+\int_0^T (1+T-t')\|(u-\tilde{u})_t(t')\|_\infty\|\tilde{u}(t')\|_{\dot{H}^2}\,dt',\end{aligned} \tag{5.22}$$

then

$$\begin{aligned}|||\mathbb{F}(u)-\mathbb{F}(\tilde{u})||| \lesssim& |1-c^2|\int_0^T (T-t')\|(u-\tilde{u})(t')\|_2\,dt' \\ &+|1-c^2|\int_0^T \|(-\Delta)^{(s-1)/2}(u-\tilde{u})(t')\|_2\,dt' \\ &+\int_0^T (1+T-t')\|(u-\tilde{u})(t')\|_{H^s}\,\|u_t(t')\|_{\dot{H}^1}\,dt' \\ &+\int_0^T (1+T-t')\|u_t(t')\|_{H^{s-1}}\|(u-\tilde{u})(t')\|_{\dot{H}^2}\,dt' \\ &+\int_0^T (1+T-t')\|\tilde{u}(t')\|_{H^s}\|(u-\tilde{u})_t(t')\|_{\dot{H}^1}\,dt' \\ &+\int_0^T (1+T-t')\|(u-\tilde{u})_t(t')\|_{H^{s-1}}\|\tilde{u}(t')\|_{\dot{H}^2}\,dt',\end{aligned} \tag{5.23}$$

therefore

$$
\begin{aligned}
|||\mathbb{F}(u)-\mathbb{F}(\tilde{u})||| \lesssim & T(1+T)|1-c^2|\|u-\tilde{u}\|_{L^\infty_T H^s} \\
& + T(1+T)\{\|u-\tilde{u}\|_{L^\infty_T H^s}\|u_t\|_{L^\infty_T H^{s-1}} \\
& + \|\tilde{u}\|_{L^\infty_T H^s}\|(u-\tilde{u})_t\|_{L^\infty_T H^{s-1}}\}.
\end{aligned} \tag{5.24}
$$

For all $u, \tilde{u} \in X_T^M$

$$
|||\mathbb{F}(u)-\mathbb{F}(\tilde{u})||| \lesssim T(1+T)(|1-c^2|+M)|||u-\tilde{u}|||. \tag{5.25}
$$

For the choice of T and M in (5.18) we have that

$$
CT(1+T)(|1-c^2|+M) < 1
$$

therefore $\mathbb{F}$ is a contraction en X_T^M.

According to Banach fixed point theorem, there exist a unique solution in X_T^M of the initial value problem (5.3). By standard arguments, we can guarantee there exists a unique solution in

$$
C([0,T); H^s(\mathbb{T}\times\mathbb{R})\times H^{s-1}(\mathbb{T}\times\mathbb{R})). \qquad \square
$$

Concluding Remarks

Here we point out some open problems connected with this work.

1. In Chapters 2 and 3, we proved that the IVP associated to the isotropic Benney-Luke (BL) and the p-generalized Benney-Luke (p-gBL) equations are globally well-posed for initial data in $\dot{H}^2(\mathbb{R}^2) \cap \dot{H}^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$. But it is not clear whether or not one can have local well-posedness results in the space $H^s(\mathbb{R}^2) \times H^{s-1}(\mathbb{R}^2)$, for $s < 2$.

2. In Chapter 2, we also proved that IVP associated to the (BL) is locally well-posed in $H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$ for $2 < s \leq 5/2$. It would be interesting to determine whether or not this result could be improved to obtain the global solutions in $\dot{H}^2(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$.

3. It is possible to prove global well-posedness for the IVP associated to the generalized Benney-Luke equation (gBL) for initial data in $\dot{H}^2(\mathbb{T} \times \mathbb{R}) \cap \dot{H}^1(\mathbb{T} \times \mathbb{R}) \times H^1(\mathbb{T} \times \mathbb{R})$ using the same techniques utilized to prove Theorem 2.1.2. This will be done somewhere else.

4. Another interesting problem regards the nonlinear scattering for the p-generalized Benney-Luke (p-gBL) equations. It seems reasonable to obtain some results in this direction since we already have global solu-

tions in the energy space and the solutions have good local regularity properties.

5. Using the theory developed here seems possible to establish local well-posedness for IVP associated to following modified Benney-Luke equation

$$\begin{aligned}\Phi_{tt} - \Delta\Phi + \mu(a\Delta^2\Phi - b\Delta\Phi_{tt}) + \epsilon(B\Delta^2\Phi_{tt} - A\Delta^3\Phi) \\ + \epsilon(\Phi_t\Delta\Phi + 2\nabla\Phi\cdot\nabla\Phi_t) = 0,\end{aligned} \tag{5.26}$$

where $\epsilon = \mu^2$ and the parameters A, B are linked. The equation (5.26) was proposed by Paumond in [25], which is still valid when we suppose that $a - b + 1/3 = \alpha$ is equal or close to $1/3$. We remind that the model given by isotropic Benney-Luke equation (1) does not hold for $a = b$ ($\alpha = 1/3$).

Appendix

Generalized Strichartz Inequalities for the Wave Equation

Notation and Preliminaries

The purpose of the present appendix is to give a self-contained form to this work and follows the ideas of an article of J.Ginibre and G.Velo, [10].

To keep the previous notation we express the standards norms in the following form:

$$\|u\|_{L_t^q X} = \|u; L^q(\mathbb{R}; X)\| := \left(\int_{\mathbb{R}} \|u(t)\|_X^q dt \right)^{1/q},$$

where X is a normed linear space.

We take the space dimension $n \geq 2$. Exponents in the spaces L^q are best characterized by positive multiples of the basic function $\alpha(q) = 1/2 - 1/q$ which have the following properties:

i) they are linear in $1/q$ and therefore behave linearly under interpolation;

ii) they vanish at $q = 2$;

iii) they are increasing in q.

Of special interest are the combinations

$$\beta(q) = \frac{n+1}{2}\alpha(q), \quad \gamma(q) = (n-1)\alpha(q), \quad \delta(q) = n\alpha(q).$$

The exponent $\beta(q)$ is the loss of derivatives in an estimate used later, the exponent $\gamma(q)$ is the optimal time decay exponent of L^q solutions of the wave equations, and $\delta(q)$ appears naturally in the Hölder and Sobolev inequalities because n/q is the degree of x of the L^q norm.

We denote convolution in x or in t by $*_x$ or $*_t$ and we define the Paley-Littlewood dyadic decomposition in the following standard form. Let $\hat{\psi} \in C_0^\infty(\mathbb{R}^n)$ with values betwen 0 and 1, $\hat{\psi}(\xi) = 1$ for $|\xi| \leq 1$ and $\hat{\psi}(\xi) = 0$ for $|\xi| \geq 2$.

We define $\hat{\varphi}_0(\xi) = \hat{\psi}(\xi) - \hat{\psi}(2\xi)$ and for any $j \in \mathbb{Z}$, $\hat{\varphi}_j(\xi) = \hat{\varphi}_0(2^{-j}\xi)$ so that

$$\operatorname{Supp} \hat{\varphi}_j \subset \{\xi : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$$

and for any $\xi \in \mathbb{R}^n \setminus \{0\}$ we have

$$\sum_{j \in \mathbb{Z}} \hat{\varphi}_j(\xi) = 1$$

with at most two nonvanishing terms in the sum. We also define $\tilde{\varphi}_j = \varphi_{j-1} + \varphi_j + \varphi_{j+1}$ for all $j \in \mathbb{Z}$. Then $\hat{\varphi}_j = \hat{\tilde{\varphi}}_j \hat{\varphi}_j$ thereby allowing for the use of the standard trick

$$\varphi_j * u = \tilde{\varphi}_j * \varphi_j * u \tag{5.27}$$

for any tempered distribution u.

The proofs of the inequalities naturally yield then in terms of Besov spaces.The Sobolev or L^q version of the inequalities follows from the Besov version by the known embeddings between those spaces. With each tempered distribution u we associate the sequence of C^∞ functions $\varphi_j * u$, to be considered as a function of two variables j and x. The homogeneous Besov spaces $\dot{B}^{\rho}_{q,s}$ is then defined for any $\rho \in \mathbb{R}$ and $1 \leq q, s \leq \infty$ by

$$\dot{B}^{\rho}_{q,s} = \{u : \|u; \| = \|2^{\rho j}\varphi_j * u; l^s_j L^q_x\| < \infty\} \tag{5.28}$$

where one takes first the L^q norm in the variable x and then the l^s norm in the variable j; i.e.

$$\|u\|_{\dot{B}^{\rho}_{q,s}} \equiv \|2^{\rho j}\varphi_j * u\|_{l^s_j L^q(\mathbb{R}^n)} = \{\sum_{j\in\mathbb{Z}} 2^{\rho j s}\|\varphi_j * u\|^s_{L^q(\mathbb{R}^n)}\}^{1/s}$$

Similarly the homogeneous Triebel-Lizorkin space $\dot{F}^{\rho}_{q,s}$ is defined, for $q < \infty$, by

$$\dot{F}^{\rho}_{q,s} = \{u : \|u; \dot{F}^{\rho}_{q,s}\| = \|2^{\rho j}\varphi_j * u; L^q_x l^s_j\| < \infty\}$$

where the norms are taken in the opposite order. By the Minkowsky inequality

$$l^s L^q \subset L^q l^s \Rightarrow \dot{B}^{\rho}_{q,s} \subset \dot{F}^{\rho}_{q,s} \qquad \text{for} \quad q \geq s$$

$$l^s L^q \supset L^q l^s \Rightarrow \dot{B}^{\rho}_{q,s} \supset \dot{F}^{\rho}_{q,s} \qquad \text{for} \quad q \leq s.$$

Comparison with the homogeneous Sobolev spaces $\dot{H}^{\rho}_q$ follows from the Paley-Littlewood theory. More precisely, from the Hilbert space valued version of the Mikhlin-Hörmander theorem which implies that

$$\dot{H}^{\rho}_q = \dot{F}^{\rho}_{q,2}$$

for all $\rho \in \mathbb{R}$ and $1 < q < \infty$, thereby yielding the inclusions

$$\dot{B}^{\rho}_{q,2} \subset \dot{H}^{\rho}_q \quad \text{for} \quad 2 < q < \infty \qquad \dot{B}^{\rho}_{q,2} \supset \dot{H}^{\rho}_q \quad \text{for} \quad 1 < q \leq 2. \tag{5.29}$$

Lemma 5.1.2. *Let* $1 \leq q_2 \leq q_1 \leq \infty$, $\rho_1, \rho_2 \in \mathbb{R}$ *with* $\rho_1 + \delta(q_1) = \rho_2 + \delta(q_2)$. *Then* $\dot{B}^{\rho_2}_{q_2,s} \subset \dot{B}^{\rho_1}_{q_1,s}$ *and*

$$\|u; \dot{B}^{\rho_1}_{q_1,s}\| \leq C\|u; \dot{B}^{\rho_2}_{q_2,s}\|.$$

Proof. See [10], page 54. □

We shall state the Strichartz inequalities in the Besov spaces version, which is both the stronger one and the easier one to prove. We will need only the Besov spaces with $s = 2$ and we will write $\dot{B}^{\rho}_{q} = \dot{B}^{\rho}_{q,2}$. The Sobolev version of the inequalities follows from the Besov version by the embedding (5.29) and is obtained by replacing everywhere $\dot{B}^{\rho}_{q}$ by $\dot{H}^{\rho}_{q}$ and excluding the cases $q = 1$ and $q = \infty$.

We are interested in the Cauchy problem for the wave equation

$$\begin{cases} u_{tt} - \Delta u = f \\ u(0, \mathbf{x}) = u_0(\mathbf{x}), \qquad u_t(0, \mathbf{x}) = u_1(\mathbf{x}). \end{cases} \tag{5.30}$$

We define the operators $U(t) = exp(i\omega t)$, $K(t) = \omega^{-1} \sin \omega t$, and $\dot{K}(t) = \cos \omega t$, with $\omega = (-\Delta)^{1/2}$. The Cauchy problem (5.30) is solved by $u = v+w$, where v is the solution of the homogeneous equation with the same data

$$\begin{cases} v_{tt} - \Delta v = 0 \\ v(0, \mathbf{x}) = u_0(\mathbf{x}), \qquad v_t(0, \mathbf{x}) = u_1(\mathbf{x}), \end{cases} \tag{5.31}$$

namely

$$v(t) = \dot{K}(t)u_0 + K(t)u_1 \qquad v_t(t) = K(t)\Delta u_0 + \dot{K}(t)u_1,$$

and w is the solution of the inhomogeneous equation with zero data,

$$\begin{cases} w_{tt} - \Delta w = f \\ w(0, \mathbf{x}) = 0, \qquad w_t(0, \mathbf{x}) = 0. \end{cases} \tag{5.32}$$

Let $L(t)$ be any of the operators $\omega^{\lambda} U(t)$, $\omega^{\lambda} K(t)$, or $\omega^{\lambda} \dot{K}(t)$, with $\lambda \in \mathbb{R}$ and let χ_+ and χ_- be the characteristic function of $\mathbb{R}_+$ and $\mathbb{R}_-$ in time. We define $L_R(t) = \chi_+(t)L(t)$ and $L_A(t) = \chi_-(t)L(t)$ where R and A stand for retarded and advanced. Then the Cauchy problem is solved for positive time by

$$\begin{cases} w(t) = \int_0^t K(t-t')f(t')dt' = (K_R *_t \chi_+ f)(t), \\ w_t(t) = \int_0^t \dot{K}(t-t')f(t')dt' = (\dot{K}_R *_t \chi_+ f)(t). \end{cases} \tag{5.33}$$

Similar formulas with advanced operators solve the Cauchy Problem 5.32 for negatives times. We restrict our attention from now on to positive times. The initial data (u_0, u_1) for the problem will be taken from the space

$$Y^\mu \equiv \dot{H}^\mu(\mathbb{R}^n) + \dot{H}^{\mu-1}(\mathbb{R}^n) \tag{5.34}$$

for $\mu \in \mathbb{R}$ arbitrary. The most studied case so far is the case $\mu = 1$ of finite energy solutions.

Statement and sketch of the proof.

We go now to state the generalized Strichartz inequalities .

Theorem 5.1.3. *Let $\rho_1, \rho_2, \mu \in \mathbb{R}$ and $2 \leq q_1, q_2, r_1, r_2 \leq \infty$ and let the following conditions be satisfied*

$$0 \leq 2/r_i \leq Min(\gamma(q_i), 1) \qquad for \quad i = 1, 2 \tag{5.35}$$

$$(2/r_i, \gamma(q_i)) \neq (1, 1) \qquad for \quad i = 1, 2 \tag{5.36}$$

$$\rho_1 + \delta(q_1) - 1/r_1 = \mu \tag{5.37}$$

$$\rho_1 + \delta(q_1) - 1/r_1 = 1 - (\rho_2 + \delta(q_2) - 1/r_2). \tag{5.38}$$

1.- Let $(u_0, u_1) \in Y^\mu$(see 5.34). Then v satisfies the estimates

$$\|v; L^{r_1}(\mathbb{R}, \dot{B}^{\rho_1}_{q_1})\| + \|v_t; L^{r_1}(\mathbb{R}, \dot{B}^{\rho_1 - 1}_{q_1})\| \leq C\|(u_0, u_1); Y^\mu\|. \tag{5.39}$$

2.- For any interval $I \subset \mathbb{R}$, possibly unbounded, the following estimates hold

$$\|K * f; L^{r_1}(I, \dot{B}^{\rho_1}_{q_1})\| \leq C\|f; L^{r'_2}(I, \dot{B}^{-\rho_2}_{q'_2})\|. \tag{5.40}$$

3.-For any interval $I = [0,T)$, $0 < T < \infty$, the function $w = K_R * \chi_+ f$ defined by (5.33) satisfies the estimates

$$\|w; L^{r_1}(I, \dot{B}^{\rho_1}_{q_1})\| + \|w_t; L^{r_1}(I, \dot{B}^{\rho_1-1}_{q_1})\| \leq C\|f; L^{r'_2}(I, \dot{B}^{-\rho_2}_{q'_2})\|. \tag{5.41}$$

The constants C are independent of I.

The same results hold with $\dot{B}^{\rho}_{q}$ replaced by $\dot{H}^{\rho}_{q}$, everywhere under the additional assumption that $q_i < \infty\, (i = 1, 2)$ whenever q_i occurs.

Sketch of the Proof. Recalling the definitions of v, w, K and $\dot{K}$, the inequalities (5.39), (5.40) and (5.41) follow from the corresponding inequalities involving only U. In addition, since the operator ω^{λ} is an isomorphism from $\dot{B}^{\rho}_{q}$ to $\dot{B}^{\rho-\lambda}_{q}$ for all $\lambda \in \mathbb{R}$, we can fix arbitrarily $\mu = 0$ without restricting the generality. It will be therefore sufficient to prove the inequalities

$$\|U(\cdot)u; L^{r_1}(\mathbb{R}, \dot{B}^{\rho_1}_{q_1})\| \leq C\|u\|_2, \tag{5.42}$$

$$\|U * f; L^{r_1}(I, \dot{B}^{\rho_1}_{q_1})\| \leq C\|f; L^{r'_2}(I, \dot{B}^{-\rho_2}_{q'_2})\|, \tag{5.43}$$

for $I \subset \mathbb{R}$ and

$$\|U_R * f; L^{r_1}(I, \dot{B}^{\rho_1}_{q_1})\| \leq C\|f; L^{r'_2}(I, \dot{B}^{-\rho_2}_{q'_2})\|. \tag{5.44}$$

for $I = [0,T) \subset \mathbb{R}^+$, under the conditions (5.35) and (5.36) and

$$\rho_i + \delta(r_i) - 1/q_i = 0 \qquad for \quad i = 1, 2. \tag{5.45}$$

The shift by one from (5.38) to (5.45) is due to the change from K to U.

We start from the estimate

$$\sup_x \left| \int exp(it|\xi| + ix\xi)\hat{\varphi}_0(\xi)d\xi \right| \leq \min\{\|\hat{\varphi}_0\|_1, C_0|t|^{-(n-1)/2}\} \tag{5.46}$$

the non-trivial part of which is a stationary phase estimate (see [11]). Scaling ξ by 2^{-j} and x, t by 2^j, we obtain

$$\sup_x \left| \int exp(it|\xi| + ix\xi)\hat{\varphi}_j(\xi)d\xi \right| \leq Min\{\|\hat{\varphi}_0\|_1 2^{nj}, C_0|t|^{-(n-1)/2}2^{j(n+1)/2}\} \tag{5.47}$$

which means that

$$\|U(t)\varphi_j\|_\infty \leq C \min\{2^{nj}, |t|^{-(n-1)/2}2^{j(n+1)/2}\}. \tag{5.48}$$

Let now f be a sufficiently regular function in the space variable. We estimate

$$\|\varphi_j * U(t)f\|_\infty = \|\varphi_j * U(t)\tilde{\varphi}_j * f\|_\infty \leq \|U(t)\varphi_j\|_\infty \|\tilde{\varphi}_j * f\|_1 \tag{5.49}$$

by (5.27) and the Young inequality, and therefore by (5.48)

$$\|\varphi_j * U(t)f\|_\infty \leq C \min\{2^{nj}, |t|^{-(n-1)/2}2^{j(n+1)/2}\}\|\tilde{\varphi}_j * f\|_1. \tag{5.50}$$

By interpolation between (5.50) and the unitarity of $U(t)$ in L^2, we obtain

$$\|\varphi_j * U(t)f\|_q \leq C \min\{2^{2j\delta(q)}, |t|^{-\gamma(q)}2^{2j\beta(q)}\}\|\tilde{\varphi}_j * f\|_{q'} \tag{5.51}$$

for $2 \leq q \leq \infty$. From now on, we consider separately the case $q > 2$ and the limiting case $q = 2$.

The case $q > 2$. We multiply (5.51) by $2^{j\beta(q)}$ and take the norm in l^2_j, thereby obtaining

$$\|U(t)f; \dot{B}_q^{-\beta(q)}\|_{l^2_j} \leq C|t|^{-\gamma(q)}\|f; \dot{B}_q'^{\beta(q)}\|, \tag{5.52}$$

where we have discarded the first term in the minimun and used the definition of $\tilde{\varphi}_j$ and the definition (5.28) of Besov spaces.

Let now f depend also on time and rewrite (5.52) as

$$\|U(t-t')f(t'); \dot{B}_q^{-\beta(q)}\| \leq C|t-t'|^{-\gamma(q)}\|f(t'); \dot{B}_q'^{\beta(q)}\|. \tag{5.53}$$

Let $0 \leq 2/q = \gamma(q) < 1$ and let $I \subset \mathbb{R}$ be an interval. Integrating over t', taking the L^r norm in time and applying the Hardy-Littlewood-Sobolev inequality, we obtain

$$\|U_R *_t f; L^r(I, \dot{B}_q^{-\beta(q)})\| \leq C\|f; L^{q'}(I, \dot{B}_{q'}^{-\beta(q)})\|, \tag{5.54}$$

where U_R stands either for U or for U_R. Now the last equation without and with retardation is the diagonal case ($r_1 = r_2$, $q_1 = q_2$) of the limiting case $2/r_i = \gamma(q_i)$.

See [10] for the end of the proof, which proceeds by abstract duality arguments.

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