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Upper and Lower Bounds for Finite $B_h[g]$ Sequences

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We give a non-trivial upper bound for $F_h(g,N)$, the size of a $B_h[g]$ subset of $\{1,\dots,N\}$, when $g>1$. In particular, we prove $F_2(g,N)\leqslant 1.864(gN)^{1/2}+1$, and $F_h(g,N)\leqslant \frac{1}{(1+\cos^h(\pi/h))^{1/h}}(hh!gN)^{1/h}$, $h>2$. On the other hand, we exhibit $B_2[g]$ subsets of $\{1,\dots,N\}$ with

$$\frac{g+[g/2]}{\sqrt{g+2[g/2]}}N^{1/2}+o(N^{1/2}),$$

elements.

1. UPPER BOUNDS

Let $h\geqslant 2$, $g\geqslant 1$ be integers. A subset A of integers is called a $B_h[g]$-sequence (or $B_h[g]$ subset) if for every positive integer m, the equation

$$m=x_1+\cdots+x_h,\qquad x_1\leqslant\cdots\leqslant x_h,\quad x_i\in A$$

has at most g distinct solutions.

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Let $F_h(g,N)$ denote the maximum size of a $B_h[g]$ sequence contained in $[1,N]$. If A is a $B_h[g]$ subset of $\{1,\dots,N\}$, then $\binom{|A|+h-1}{h} \leqslant ghN$, which implies the trivial upper bound

$$F_h(g,N) \leqslant (ghh!N)^{1/h}. \tag{1.1}$$

For $g=1$, $h=2$, it is possible to take advantage of counting the differences $x_i - x_j$ instead of the sums $x_i + x_j$, because the differences are also distinct. In this way, Erdős and Turán [2] proved that $F_2(1,N) \leqslant N^{1/2} + O(N^{1/4})$, which is the best possible except for the estimate of error term.

For $h = 2m$, Jia [4] proved $F_{2m}(1,N) \leqslant (m(m!)^2)^{1/2m} N^{1/2m} + O(N^{1/4m})$. A similar upper bound for $F_{2m-1}(1,N)$ has been proved independently by Chen [1] and Graham [3]: $F_{2m-1}(1,N) \leqslant ((m!)^2)^{1/2m-1} N^{1/2m-1} + O(N^{1/4m-2})$.

However, for $g > 1$, the situation is completely different because the same difference can appear many times, and for $g > 1$ nothing better than (1.1) is known. In this paper, we improve this trivial upper bound.

THEOREM 1.1. *Given positive integers h, g and N, we have*

$$F_2(g,N) \leqslant 1.864(gN)^{1/2} + 1,$$

and

$$F_h(g,N) \leqslant \frac{1}{(1+\cos^h(\pi/h))^{1/h}} (hh!gN)^{1/h}, \qquad \text{when } h > 2.$$

Proof. Let $A \subset [1,N]$ be a $B_h[g]$ sequence with $|A| = k$. Put $f(t) = \sum_{a\in A} e^{iat}$. We have $f(t)^h = \sum_{n=h}^{hN} r_h(n)e^{int}$ where $r_h(n)$ is the number of solutions of $n = a_1 + \cdots + a_h$, $a_i \in A$ $(i = 1,2,\dots,h)$. Then we can write

$$f(t)^h = h!g \sum_{n=h}^{hN} e^{int} - \sum_{n=h}^{hN} (h!g - r_h(n))e^{int} = h!gp(t) - q(t).$$

Since $r_h(n) \leqslant h!g$, we have

$$\begin{aligned}\sum_{n=h}^{hN} |h!g - r_h(n)| &= \sum_{n=h}^{hN} (h!g - r_h(n)) \\ &= (h(N-1)+1)h!g - \sum r_h(n) \\ &= (h(N-1)+1)h!g - k^h,\end{aligned}$$

thus $|q(t)| \leqslant hh!gN - k^h$ for every value of t. The series $p(t)$ is just a geometrical series and we can express it as

$$p(t) = e^{hit}\frac{1 - e^{i(h(N-1)+1)t}}{1 - e^{it}}$$

if $0 < t < 2\pi$. We shall use only the property that at values of the form $t = jt_h$, $t_h = \frac{2\pi}{h(N-1)+1}$ with integer j, $1 \leqslant j \leqslant h(N-1)$, we have $p(t) = 0$, thus $f(t)^h = q(t)$. Consequently, $|f(jt_h)| \leqslant (hh!gN - k^h)^{1/h}$ for any integer j, $1 \leqslant j \leqslant h(N-1)$.

Since the midpoint of the interval $[1, N]$ is $(N+1)/2$, it will be useful to express f as

$$f(t) = e^{((N+1)/2)it} f^*(t),$$

where

$$f^*(t) = \sum_{a \in A} e^{(a-((N+1)/2))it}.$$

Now we consider a function $F(x) = \sum_{j=1}^{h(N-1)} b_j \cos(jx)$ satisfying $F(x) \geqslant 1$ for $|x| \leqslant \pi/h$. We define $C_F = \sum |b_j|$.

We are looking for a lower and an upper bound for $\Re\left(\sum_{j=1}^{h(N-1)} b_j f^*(jt_h)\right)$.

$$\begin{aligned}\Re\left(\sum_{j=1}^{h(N-1)} b_j f^*(jt_h)\right) &\leqslant \sum_{j=1}^{h(N-1)} |b_j||f^*(jt_h)| \\ &= \sum_{j=1}^{h(N-1)} |b_j||f(jt_h)| \\ &\leqslant \left(\sum_{j=1}^{h(N-1)} |b_j|\right)(hh!gN - k^h)^{1/h} \\ &= C_F(hh!gN - k^h)^{1/h}. \qquad (1.2)\end{aligned}$$

On the other hand,

$$\begin{aligned}\Re\left(\sum_{j=1}^{h(N-1)} b_j f^*(jt_h)\right) &= \Re\left(\sum_{a \in A}\sum_{j=1}^{h(N-1)} b_j e^{i(a-((N-1)/2))t_h j}\right) \\ &= \sum_{a \in A}\sum_{j=1}^{h(N-1)} b_j \cos\left(\left(a - \frac{N-1}{2}\right)t_h j\right) \\ &= \sum_{a \in A} F\left(\left(a - \frac{N-1}{2}\right)t_h\right) \geqslant k, \qquad (1.3)\end{aligned}$$

because $|(a-\frac{N-1}{2})t_h|\leqslant\pi/h$ for any integer $a\in A$. From (1.2) and (1.3) we have

$$|A|=k\leqslant\frac{1}{\left(1+\frac{1}{C_F^h}\right)^{1/h}}(hh!gN)^{1/h}.$$

For $h>2$, we can take $F(x)=\frac{1}{\cos(\pi/h)}\cos(x)$, which satisfies the conditions above with $C_F=\frac{1}{\cos(\pi/h)}$ and this proves the theorem for $h>2$. For $h=2$, we can take $F(x)=2\cos(x)-\cos(2x)$, $C_F=3$, which gives $|A|\leqslant\frac{6}{\sqrt{10}}\sqrt{gN}$, a non-trivial upper bound. However, an infinite series gives a better result. Take the function

$$F(x)=\begin{cases}1, & |x|\leqslant\pi/2,\\ 1+\pi\cos(x), & \pi/2<|x|\leqslant\pi.\end{cases}$$

It is easy to see that $F(x)=\frac{\pi}{2}\cos(x)+2\sum_{n=2}^{\infty}\frac{\cos(\pi n/2)}{n^2-1}\cos(nx)$. This series satisfies that $F(x)=1$ for $|x|\leqslant\pi/2$ with

$$C_F=\pi/2+2\sum_{n=1}^{\infty}\frac{1}{4n^2-1}=\pi/2+2\sum_{n=1}^{\infty}\frac{1}{2}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)=\pi/2+1.$$

However, we must truncate the series to the integers $n\leqslant 2(N-1)$. Let

$$F_T(x)=\frac{\pi}{2}\cos(x)+2\sum_{n=2}^{2(N-1)}\frac{\cos(\pi n/2)}{n^2-1}\cos(nx).$$

Observe that $|F_T(x)-F(x)|\leqslant 2\sum_{2N-1}^{\infty}\frac{1}{n^2-1}=\frac{1}{2N-2}$. Now we consider the polynomial $F^*(x)=\frac{2N-2}{2N-3}F_T(x)$. If $|x|<\pi/2$ we have

$$\begin{aligned}|F^*(x)|&=\frac{2N-2}{2N-3}|F(x)+F_T(x)-F(x)|\\ &\geqslant\frac{2N-2}{2N-3}(|F(x)|-|F_T(x)-F(x)|)\\ &\geqslant\frac{2N-2}{2N-3}\left(1-\frac{1}{2N-2}\right)\\ &=1,\end{aligned}$$

and $C_{F^*}\leqslant\frac{2N-2}{2N-3}(\pi/2+1)$. Thus

$$|A|=k\leqslant\frac{2}{\left(1+\frac{1}{C_{F^*}^2}\right)^{1/2}}(gN)^{1/2}.$$

A simple calculation gives

$$\frac{2}{\left(1+\frac{1}{C_{F^*}^2}\right)^{1/2}}-\frac{2}{\left(1+\frac{1}{C_F^2}\right)^{1/2}}\leqslant\frac{1}{N}.$$

Then

$$|A|=k\leqslant\frac{2}{\left(1+\frac{1}{C_F^2}\right)^{1/2}}(gN)^{1/2}+\sqrt{\frac{g}{N}}$$
$$=\frac{2\pi+4}{\sqrt{\pi^2+4\pi+8}}\sqrt{gN}+\sqrt{\frac{g}{N}}\leqslant 1.864\sqrt{gN}+1$$

because, obviously, $g\leqslant N$. ∎

2. LOWER BOUNDS

Now we are interested in finite $B_2[g]$ sequences as dense as possible. Kolountzakis [6] exhibits a $B_2[2]$ subset of $\{1,\ldots,N\}$ with $\sqrt{2}N^{1/2}+o(N^{1/2})$ elements taking $A=(2A_0)\cup(2A_0+1)$ with A_0, a $B_2[1]$ sequence contained in $\{1,\ldots,[N/2]\}$.

In general it is easy to construct a $B_2[g]$ subset of $\{1,\ldots,N\}$ with $(gN)^{1/2}+o(N^{1/2})$ elements. In the sequel we improve these results.

THEOREM 2.1.

$$F_2(g,n)\geqslant\frac{g+[g/2]}{\sqrt{g+2[g/2]}}N^{1/2}+o(N^{1/2}). \tag{2.1}$$

For $g=2$, Theorem 2.1 gives

$$F_2(2,N)\geqslant\tfrac{3}{2}N^{1/2}+o(N^{1/2}).$$

In general, for g even we get

$$F_2(g,N)\geqslant\frac{3}{2\sqrt{2}}(gN)^{1/2}+o(N^{1/2}).$$

And for g odd,

$$F_2(g,N)\geqslant\frac{3-(1/g)}{2\sqrt{2-(1/g)}}(gN)^{1/2}+o(N^{1/2}).$$

Remark. Jia's constructions of $B_h[g]$ sequences in [5] does not work (Jia, personal communication). In the last step of the proof of Theorem 3.1 of Jia [5] we cannot deduce from the hypothesis that $\{b_{s1}, \ldots, b_{sh}\} = \{b_{t1}, \ldots, b_{th}\}$. Jia's argument can be modified if we define $g_a(h, m)$ as the number of solutions of the equation $a \equiv x_1 + \cdots + x_h \pmod m$, $0 \leqslant x_i \leqslant m-1$. It would imply the result $|B| = \sqrt{gN} + o(\sqrt{N})$. But for $g = 2$ it is the Kolountzakis's construction [6].

We need some definitions and lemmas in order to construct $B_2[g]$ sequences satisfying Theorem 2.1.

Definition 2.2. We say that $a_0, a_1, \ldots, a_k$ satisfies the $B^*[g]$ condition if the equation $a_i + a_j = r$ has at most g solutions. (Here, $a_i + a_j = a_j + a_i$ counts as two solutions if $i \neq j$.)

Definition 2.3. We say that a sequence of integers C is a $B_2 \pmod m$ sequence if $c_i + c_j \equiv c_k + c_l \pmod m$ implies $\{c_i, c_j\} = \{c_k, c_l\}$.

Lemma 2.4. *If $a_0, a_1, \ldots, a_k$ satisfies the $B^*[g]$ condition, and C is a $B_2 \pmod m$ sequence, then the sequence $B = \bigcup_{i=0}^{k} (C + ma_i)$ is a $B_2[g]$ sequence.*

Proof. If $b_1 + b'_1 = b_2 + b'_2 = \cdots = b_{g+1} b'_{g+1}$, $b_j, b'_j \in B$ for $j = 1, \ldots, g+1$, we may write

$$b_j = c_j + a_{i_j} m, \quad b'_j = c'_j + a'_{i_j} m,$$

where $c_j, c'_j \in C$ and $a_{i_j}, a'_{i_j} \in \{a_0, \ldots, a_k\}$, and b_j, b'_j are ordered so that $c_j \leqslant c'_j$.

First, we note that

$$c_j + c'_j \equiv c_1 + c'_1 \pmod m \quad \text{for all } j\text{'s},$$

which implies $c_j = c_1$, and $c'_j = c'_1$ for all $j = 1, \ldots, g+1$ because C is a $B_2 \pmod m$ sequence. Therefore, all the $g+1$ sums $a_{i_j} + a_{i'_j}$ are equal. Since $\{a_0, \ldots, a_k\}$ satisfies the $B^*[g]$ condition, we see that there exist μ and v with $\mu \neq v$ such that

$$a_{i_\mu} = a_{i_v} \qquad \text{and} \qquad a'_{i_\mu} = a'_{i_v},$$

which implies that $b_\mu = b_v$ and $b'_\mu = b'_v$. ∎

LEMMA 2.5. *The subset*

$$A^g = A_1^g \cup A_2^g = \{k;\ 0 \leqslant k \leqslant g-1\} \cup \{g-1+2k;\ 1 \leqslant k \leqslant [g/2]\}$$

satisfies the condition $B^*[g]$.

Proof. Let $r(m) = |\{a;\ a, m-a \in A^g\}|$ and $r_{ij}(m) = |\{a;\ a \in A_i^g, m-a \in A_j^g\}|$, $1 \leqslant i,\ j \leqslant 2$. We have $r(m) = r_{11}(m) + 2r_{12}(m) + r_{22}(m)$ because $r_{12} = r_{21}$.

With this notation we will prove that $r(m) \leqslant g$ for any integer m. First we study the functions r_{ij}.

- $r_{11}(m)$: If $a, m-a \in A_1^g$, then $0 \leqslant a \leqslant g-1$ and $0 \leqslant m-a \leqslant g-1$, which implies

$$\max\{0, m-g+1\} \leqslant a \leqslant \min\{g-1, m\}.$$

Then

$$r_{11}(m) = \max\{0, \min\{g-1, m\} - \max\{0, m-g+1\} + 1\},$$

and

$$r_{11}(m) = \begin{cases} m+1, & 0 \leqslant m \leqslant g-1, \\ 2g-m-1, & g \leqslant m \leqslant 2g-1, \\ 0, & 2g-1 \leqslant m. \end{cases}$$

- $r_{12}(m)$: If $a \in A_2^g$, $m-a \in A_1^g$, then $a = g-1+2k$, $1 \leqslant k \leqslant [g/2]$ and

$$0 \leqslant m - (g-1+2k) \leqslant g-1,$$

which implies

$$\max\left\{1, \frac{m-2g+2}{2}\right\} \leqslant k \leqslant \min\left\{\left[\frac{g}{2}\right], \frac{m-g+1}{2}\right\}.$$

Since the k's are integers, we can write

$$\max\left\{1, \left\lceil\frac{m-2g+3}{2}\right\rceil\right\} \leqslant k \leqslant \min\left\{\left[\frac{g}{2}\right], \left[\frac{m-g+1}{2}\right]\right\}.$$

Then

$$r_{12}(m) = \begin{cases} 0 & \text{if } m \leqslant g, \\ [\frac{m-g+1}{2}] & \text{if } g \leqslant m \leqslant 2g-1, \\ [\frac{g}{2}] - [\frac{m-2g+1}{2}] & \text{if } 2g \leqslant m \leqslant 3g-1, \\ 0 & \text{if } 3g-1 \leqslant m. \end{cases}$$

- $r_{22}(m)$: Obviously, if m is odd then $r_{22}(m) = 0$.

If $a, m-a \in A_2^g$, then $a = g-1+2k$, $m-a = g-1+2j$, $1 \leqslant j$, $k \leqslant [g/2]$. We have

$$1 \leqslant j = \frac{m}{2} - (g-1) - k \leqslant \left[\frac{g}{2}\right],$$

which implies, if m is even, that

$$\max\left\{1, \frac{m}{2} - g - \left[\frac{g}{2}\right] + 1\right\} \leqslant k \leqslant \min\left\{\frac{m}{2} - g, \left[\frac{g}{2}\right]\right\}.$$

Then

$$r_{22}(m) = \max\left\{0, \min\left\{\frac{m}{2} - g, \left[\frac{g}{2}\right]\right\} - \max\left\{1, \frac{m}{2} - g - \left[\frac{g}{2}\right] + 1\right\} + 1\right\}.$$

Therefore, if m is even

$$r_{22}(m) = \begin{cases} 0 & \text{if } m < 2g, \\ m/2 - g & \text{if } 2g \leqslant m \leqslant 3g-1, \\ g + 2[g/2] - m/2 & \text{if } 3g \leqslant m \leqslant 4g-2, \\ 0 & \text{if } 4g-2 < m. \end{cases}$$

Now, we are ready to calculate $r(m)$.

- If $m \leqslant g-1$, $r(m) = r_{11}(m) = m+1 \leqslant g$.
- If $g \leqslant m \leqslant 2g-1$, $r(m) = r_{11}(m) + 2r_{12}(m) = 2g - m - 1 + 2[\frac{m-g+1}{2}] \leqslant 2g - m - 1 + m - g + 1 = g$.
- If $2g \leqslant m \leqslant 3g-1$, and m is odd, $r(m) = 2r_{12}(m) = 2([g/2] - [\frac{m-2g+1}{2}]) \leqslant g$. If m is even, $r(m) = 2r_{21}(m) + r_{22}(m) = 2([g/2] - [\frac{m-2g+1}{2}]) + m/2 - g = 2[g/2] - (m-2g) + m/2 - g = 2[g/2] + g - m/2 \leqslant 2[g/2] + g - (2g)/2 \leqslant g$.
- If $3g \leqslant m \leqslant 4g-2$, and m is odd, $r(m) = 0$. If m is even, $r(m) = r_{22}(m) = g + 2[g/2] - m/2 \leqslant g + 2[g/2] - (3g)/2 \leqslant g/2 < g$. ∎

Proof of Theorem 2.1. It is known [2], that for $m = p^2 + p + 1$, p prime, there exists a $B_2 \pmod{m}$ sequence C_m such that $|C_m| = p + 1$ and $C_m \subset [1, m]$.

Let us take

$$B = \bigcup_{i=0}^{k} (C_m + ma_i),$$

where $A^g = \{a_0, a_1, \ldots, a_k\}$ is defined in Lemma 2.5.

Observe that $B \subset [1, m(1 + a_k)]$, where $a_k = g - 1 + 2[g/2]$. Observe, also, that $|B| = |A^g||C_m| = (g + [g/2])(p + 1)$. Then, Lemma 2.4 implies that

$$F_2[g, m(g + 2[g/2])] \geqslant (g + [g/2])(p + 1).$$

For any integer n we can choose a prime p such that

$$n - o(n) \leqslant (p^2 + p + 1)(g + 2[g/2]) \leqslant n.$$

Then

$$\begin{aligned} F_2[g, n] &\geqslant F_2[g, m(g + 2[g/2])] \\ &\geqslant (g + [g/2])(p + 1) \\ &\geqslant \frac{g + [g/2]}{\sqrt{g + 2[g/2]}} n^{1/2} + o(n^{1/2}). \quad \blacksquare \end{aligned}$$

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